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Common Fixed Points for Multifunctions Satisfying a Polynomial Inequality

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Abstract

Common fixed point theorems in complete metric spaces (X, d) are given for two or more multifunctions which satisfy polynomial inequalities using only the distance d, without using the Hausdorff metric.

Keywords: *multifunctions, common fixed point*

Introduction

The paper of B. Fisher [1] contains the following result:

Theorem A. Let (X,d) be a complete metric space and $S,T: X \to X$ two mappings such that

$$d(Sx,Ty) \le c\frac{d^2(x,Sx)+d^2(y,Ty)}{d(x,Sx)+d(y,Ty)} \quad (1)$$

for all x, y from X which verify the condition $d(x,Sx)+d(y,Ty) \ne 0$, where $0<c<1$.

Then S and T have a common fixed point that is there exists $u \in X$ such that $u = Su$ and $u = Tu$.

Remark. The mappings S and T which verify theorem A have the property:

$$\text{(a)} \quad \begin{cases} \text{for any convergent sequence } (x_n)_{n\ge 0} \text{ from } X \text{ with } \lim\limits_{n\to\infty} x_n = x, \\ x_{2n+1} = Sx_{2n},\, x_{2n} = Tx_{2n-1} \text{ it results } x = Sx,\, x = Tx. \end{cases}$$

Proof. Let us assume that $d(x,Tx) \ne 0$. Based on condition (1) we have

$$d(x_{2n+1},Tx) = d(Sx_{2n},Tx) \le c\frac{d^2(x_{2n},Sx_{2n})+d^2(x,Tx)}{d(x_{2n},Sx_{2n})+d(x,Tx)} = c\frac{d^2(x_{2n},x_{2n+1})+d^2(x,Tx)}{d(x_{2n},x_{2n+1})+d(x,Tx)},$$

from where, for $n \to \infty$, it results $d(x,Tx) \le c\,d(x,Tx)$ which is a contradiction since $0<c<1$. Therefore $d(x,Tx)=0$, that is $x = Tx$. Analogously we prove that $x = Sx$.

In the following we will consider mappings, called multifunctions, defined on the metric space (X,d) with values in $P(X)$, that is in the family of nonempty subsets of X.

In [4] V. Popa has proved common fixed point theorems for multifunctions which verify rational inequalities, which contain the Hausdorff metric in their expressions and which generalize theorem A.

In the present paper we will present other common fixed point theorems for two or more multifunctions without using the Hausdorff metric and which generalize not only theorem A but also the theorems obtained by V. Popa [4] for the case $p = 2m$.

Common Fixed Point Theorems for Multifunctions

Fixed point of the multifunction $T:(X,d)\to P(X)$ is any element $u\in X$ with the property $u\in Tu$. We note $F(T)$ the set of fixed points of the multifunction T .

Lemma 1. Let (X,d) be a metric space and $S,T:X\to P(X)$ two multifunctions such that $(\forall)x\in X$, $(\forall)y\in Sx$ (or $y\in Tx$) there exists $z\in Ty$ (respectively $z\in Sy$) , the following inequality occurs:

$$(1-c)d^{2m}(y,z)+d^{m}(x,y)d^{m}(y,z)-c\,d^{2m}(x,y)\le 0 \tag{2}$$

where $m\ge 1$, $0<c<1$ and $F(S)\ne\phi$. Then $F(T)\ne\phi$ and $F(S)=F(T)$.

Proof. Let $u\in F(S)$, that is $u\in Su$, it results that there exists $z\in Tu$ and (2) becomes

$$(1-c)d^{2m}(u,z)+d^{m}(u,u)d^{m}(u,z)-c\,d^{2m}(u,u)\le 0$$

from where we get $(1-c)d^{2m}(u,z)\le 0$, that is $d(u,z)=0$. It results $z=u$ and therefore $u\in Tu$ which implies $F(S)\subset F(T)$. Analogously we prove that $F(T)\subset F(S)$. Therefore $F(S)=F(T)$.

Let $V:X\to P(X)$ with (X,d) a metric space. The following property will be used further:

$$\text{(b)}\quad \begin{cases} \text{for any convergent sequence } (x_n)_{n\ge 0} \text{ from } X \text{ with } \lim\limits_{n\to\infty} x_n = x, \\ x_{2n-1}\in Vx_{2n-2} \text{ (or } x_{2n}\in Vx_{2n-1}) \text{ it results } x\in Vx. \end{cases}$$

Theorem 1. Let (X,d) be a complete metric space and $S,T:X\to P(X)$ two multifunctions such that $(\forall)x\in X$, $(\forall)y\in Sx$ (or $y\in Tx$) there exists $z\in Ty$ (respectively $z\in Sy$) inequality (2) occurs, where $m\ge 1$, $0<c<1$. If one of the multifunctions S , T verifies condition (b) then S and T have common fixed points and $F(S)=F(T)$.

Proof. Let $x_0\in X$ arbitrary fixed and $x_1\in Sx_0$. Then there exists $x_2\in Tx_1$ such that

$$(1-c)d^{2m}(x_1,x_2)+d^{m}(x_0,x_1)d^{m}(x_1,x_2)-c\,d^{2m}(x_0,x_1)\le 0\,.$$

Then there exists $x_3\in Sx_2$ such that

$$(1-c)d^{2m}(x_2,x_3)+d^{m}(x_1,x_2)d^{m}(x_2,x_3)-c\,d^{2m}(x_1,x_2)\le 0\,.$$

Continuing this reasoning we obtain a sequence

$$x_0,\ x_1,\ x_2,\ x_3,\ \dots,\ x_{n-1},\ x_n,\dots$$

with $x_{2n-1}\in Sx_{2n-2}$, $x_{2n}\in Tx_{2n-1}$ and which verifies the inequality

$$(1-c)d^{2m}(x_n,x_{n+1})+d^{m}(x_{n-1},x_n)d^{m}(x_n,x_{n+1})-c\,d^{2m}(x_{n-1},x_n)\le 0\,,\ (\forall)n\ge 1\,. \tag{3}$$

The first member from inequality (3) is a second degree trinomial in the variable $d^m(x_n, x_{n+1})$ with the discriminant

$$\Delta = d^{2m}(x_{n-1}, x_n) + 4(1-c)c\, d^{2m}(x_{n-1}, x_n) = (1+4c-4c^2)d^{2m}(x_{n-1}, x_n) > 0 .$$

Inequality (3) occurs if $d^m(x_n, x_{n+1})$ is between the roots of the trinomial, that is

$$\frac{-1-\sqrt{1+4c-4c^2}}{2(1-c)} d^m(x_{n-1}, x_n) \le d^m(x_n, x_{n+1}) \le \frac{-1+\sqrt{1+4c-4c^2}}{2(1-c)} d^m(x_{n-1}, x_n) .$$

We note

$$k^m = \frac{-1+\sqrt{1+4c-4c^2}}{2(1-c)} .$$

A simple calculation shows that $k < 1$ and since $d(x_n, x_{n+1}) \ge 0$ it results

$$0 \le d^m(x_n, x_{n+1}) \le k^m d^m(x_{n-1}, x_n) ,$$

that is $d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n)$, $(\forall)n \ge 1$, from where we deduce

$$d(x_n, x_{n+1}) \le k^n d(x_0, x_1) ,\ (\forall)n \ge 1 .$$

A routine calculation leads to

$$d(x_n, x_{n+p}) \le \frac{k^n}{1-k} d(x_0, x_1) ,\ n, p \in \mathbf{N}^* ,$$

which shows that $(x_n)_{n\ge 0}$ it is a Cauchy sequence and since the space X is complete it results that $(x_n)_{n\ge 0}$ is convergent. Let $u = \lim_{n\to\infty} x_n$, $u \in X$.

We have $x_{2n-1} \in Sx_{2n-2}$ and assuming that S verifies (b) it results that $u \in Su$.

With lemma 1 we deduce that $u \in Tu$ and $F(S) = F(T)$.

Corollary. Theorem 1 generalizes theorem A of B. Fisher [1].

Proof. We assume that the conditions of theorem A are true. Eliminating the denominator, (1) becomes

$$d(Sx,Ty)d(x,Tx) + d(Sx,Ty)d(y,Ty) - c\, d^2(x,Sx) - c\, d^2(y,Ty) \le 0 . \tag{4}$$

We observe that (4) occurs for any $x, y \in X$, even if $d(x,Sx) + d(y,Ty) = 0$. We consider $y = Sx$ and note $z = Ty$. Inequality (4) becomes

$$(1-c)d^2(y,z) + d(x,y)d(y,z) - c\, d^2(x,y) \le 0 ,$$

which is the same with (2) for $m = 1$ and $S, T : X \to X$.

Based on this remark occurs (a) which covers (b) in this particular case. Theorem A is this way proven.

In the particular case $S = T : X \to P(X)$ from theorem 1 it results

Theorem 2. Let (X, d) be a complete metric space and $T : X \to P(X)$ a multifunction such that $(\forall)x \in X$, $(\forall)y \in Tx$, there exists $z \in Ty$ with the property

$$(1-c)d^{2m}(y,z)+d^{m}(x,y)d^{m}(y,z)-c\,d^{2m}(x,y)\leq 0,$$

where $m\geq 1$, $0<c<1$. Then T has a fixed point.

Theorem 3. Let (X,d) be a complete metric space and $(T_n)_{n\geq 1}$ a sequence of multifunctions $T_n : X\to P(X)$ such that for any $n\geq 2$ occurs the property $(\forall)x\in X$, $(\forall)y\in T_1x$ (or $y\in T_nx$), there exists $z\in T_ny$ (respectively $z\in T_1y$) which verify the condition

$$(1-c)d^{2m}(y,z)+d^{m}(x,y)d^{m}(y,z)-c\,d^{2m}(x,y)\leq 0,$$

where $m\geq 1$, $0<c<1$. If one of the multifunctions T_n verifies (b) then the sequence $(T_n)_{n\geq 1}$ has common fixed points and $F(T_1)=F(T_n)$, $(\forall)n\geq 2$.

Theorem 3 results from theorem 1 and lemma 1.

Other Consequences of Theorem 1

In this section we will deduce theorems 2, 3 and 5 from [4] (V. Popa) for the particular case $p=2m$, like a consequence of theorem 1 from this paper.

Let (X,d) be a metric space, $P_{clb}(X)$ the family of nonempty subsets, closed and bounded from X and the Hausdorff-Pompeiu metric

$$H(A,B)=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\right\}$$

with $d(a,B)=\inf\limits_{b\in B}d(a,b)$, where $A,B\in P_{clb}(X)$. We also note

$$\delta(A,B)=\sup\{d(a,b):a\in A,b\in B\}.$$

Particularizing the well known result (lemma 1 (V) [5]) which says that if $A,B\in P_{clb}(X)$ and $k\in\boldsymbol{R}$, $k>1$, then for any $a\in A$ there exists $b\in B$ such that $d(a,b)\leq kH(A,B)$, we obtain

Lemma 2. Let $k>1$ and the multifunctions $S,T:X\to P_{clb}(X)$. Then for any $x\in X$ and any $y\in Sx$ (or $y\in Tx$) there exists $z\in Ty$ (respectively $z\in Sy$) such that

$$d(y,z)\leq kH(Sx,Ty).$$

Theorem 4 (Theorem 2 [4]). Let (X,d) be a complete metric space and $T_1,T_2:X\to P_{clb}(X)$ two multifunctions such that

$$H^{m}(T_1x,T_2y)\leq c\frac{d^{2m}(x,T_1x)+d^{2m}(y,T_2y)}{\delta^{m}(x,T_1x)+\delta^{m}(y,T_2y)} \tag{5}$$

for any x, y from X for which

$$\delta^{m}(x,T_1x)+\delta^{m}(y,T_2y)\neq 0, \tag{6}$$

where $m\geq 1$, $0<c<1$. Then T_1 and T_2 have common fixed points and $F(T_1)=F(T_2)$.

Proof. Eliminating the denominator, (5) becomes

$$H^{m}(T_1x,T_2y)\left(\delta^{m}(x,T_1x)+\delta^{m}(y,T_2y)\right)\leq c\left(d^{2m}(x,T_1x)+d^{2m}(y,T_2y)\right) \tag{7}$$

which occurs even if condition (6) is not satisfied.

Inequality (7) is valid for any x, y from X and in particular for $y \in T_1 x$.

Let $1 < k < c^{-\frac{1}{m}}$. For $x \in X$, $y \in T_1 x$ with lemma 2 it results that there exists $z \in T_2 y$ such that

$$d(y,z) \le kH(T_1 x, T_2 y)$$

and from here we have

$$d^m(y,z)\left(\delta^m(x,T_1x)+\delta^m(y,T_2y)\right) \le k^m H^m(T_1x,T_2y)\left(\delta^m(x,T_1x)+\delta^m(y,T_2y)\right)$$

and now with (7) we obtain

$$d^m(y,z)\left(\delta^m(x,T_1x)+\delta^m(y,T_2y)\right) \le ck^m\left(d^{2m}(x,T_1x)+d^{2m}(y,T_2y)\right)$$

or even more

$$d^m(y,z)\left(d^m(x,y)+d^m(y,z)\right) \le ck^m\left(d^{2m}(x,y)+d^{2m}(y,z)\right),$$

from where it results that $(\forall)x \in X$, $(\forall)y \in T_1 x$, there exists $z \in T_2 y$ such that

$$(1-ck^m)d^{2m}(y,z)+d^m(x,y)d^m(y,z)-ck^m\, d^{2m}(x,y) \le 0,$$

where $m \ge 1$, $0 < ck^m < 1$, condition which has the form of inequality (2).

We prove now that T_1 verifies condition (b).

Let $(x_n)_{n\ge 0}$ be a convergent sequence from X with $\lim\limits_{n\to\infty} x_n = x \in X$ and

$$x_{2n-1} \in T_1 x_{2n-2},\ x_{2n} \in T_2 x_{2n-1}.$$

We have

$$d(T_1x, x_{2n}) \le H(T_1x, T_2x_{2n-1})$$

from where with (7) we obtain

$$d^m(T_1x,x_{2n})\left(\delta^m(x,T_1x)+\delta^m(x_{2n-1},T_2x_{2n-1})\right) \le c\left(d^{2m}(x,T_1x)+d^{2m}(x_{2n-1},T_2x_{2n-1})\right)$$

or more

$$d^m(T_1x,x_{2n})\left(d^m(x,T_1x)+d^m(x_{2n-1},x_{2n})\right) \le c\left(d^{2m}(x,T_1x)+d^{2m}(x_{2n-1},x_{2n})\right)$$

from where, for $n\to\infty$, it results $d(T_1x,x) \le c\,d(x,T_1x)$, that is $d(T_1x,x)=0$. Because T_1x it is a closed set we deduce $x \in T_1 x$.

It results that the conditions of theorem 1 are satisfied and together with lemma 1 results theorem 4.

Theorem 5 (Theorem 3 [4]). Let (X,d) be a complete metric space and $T: X \to P_{clb}(X)$ a multifunction such that the following inequality occurs

$$H^m(Tx,Ty) \le c\frac{d^{2m}(x,Tx)+d^{2m}(y,Ty)}{\delta^m(x,Tx)+\delta^m(y,Ty)}$$

for all x, y in X with $\delta^m(x,Tx)+\delta^m(y,Ty) \ne 0$. Then T has a fixed point.

This theorem results with theorem 4.

Theorem 6 (Theorem 5 [4]). Let (X,d) be a complete metric space and $(T_n)_{n\geq 1}$ a sequence of multifunctions $T_n : X \to P_{clb}(X)$ such that the following inequality occurs

$$H^m(T_1x,T_ny) \leq c\frac{d^{2m}(x,T_1x)+d^{2m}(y,T_ny)}{\delta^m(x,T_1x)+\delta^m(y,T_ny)},\ (\forall)n \geq 2$$

for all x, y in X which verify the condition $\delta^m(x,T_1x)+\delta^m(y,T_ny) \neq 0$, where $m \geq 1$, $0 < c < 1$.

Then the sequence $(T_n)_{n\geq 1}$ has common fixed points and $F(T_1) = F(T_n)$, $(\forall)n \geq 2$.

This theorem results with theorem 1 and lemma 1.

Note. In paper [2] fixed point theorems in metric spaces (X,d) are given for multifunctions $T : X \to P(X)$, called (d)-contractive, without using the Hausdorff metric, having the property

$(\forall)x \in X$, $(\forall)y \in Tx$, there exists $z \in Ty$ such that $d(y,z) \leq \alpha\, d(x,y)$, where $0 < \alpha < 1$. See also [3] for related results on common fixed points theorems for (d)-contractive multifunctions.

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Puncte fixe commune pentru multifuncții care verifică o inegalitate polinomială

Rezumat

Se dau teoreme de punct fix comun în spații metrice complete (X,d) *pentru două sau mai multe multifuncții care îndeplinesc inegalități polinomiale exprimate numai cu distanța* d*, fără a utiliza metrica Hausdorff.*