BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LXI No. 2/2009	11 - 20	Seria Matematică - Informatică - Fizică

The Hilbert Function of the Intersection of Two Base Rings Associated to Some Transversal Polymatroids

Nicolae-Alin Ştefan

Petroleum –Gas University of Ploieşti, Mathematics Department, Bd. Bucureşti, 39, Ploieşti
e-mail: nastefan@upg-ploiesti.ro

Abstract

The intersection of two base rings associated to some transversal polymatroids is not necessary the base ring associated to a transversal polymatroid. In this paper, we compute the Hilbert series and the Hilbert function when the intersection of such base ring is the base ring associated to a transversal polymatroid.

Key words: *Base ring, transversal polymatroid, equations of a cone, Hilbert function, Hilbert series*

Introduction

The discrete polymatroids and their base rings are studied recently in many papers (see [3], [4], [5], [6], [7], [8], [10], [11]). It is important to give conditions when the base ring associated to a transversal polymatroid is Gorenstein (see [3]). In [5] we introduced a class of such base rings and in [6] we note that an intersection of such base rings (introduced in [5]) is Gorenstein and give necessary and sufficient conditions for the intersections of two base rings from [5] to be still a base ring of a transversal polymatroid. In this paper we compute the Hilbert function of the intersections of two base rings from [5] which is still a base ring of a transversal polymatroid.

Preliminaries

We use the symbols $\boldsymbol{Z}$, $\boldsymbol{R}$ to be the integral and real numbers. The subset of nonnegative elements in $\boldsymbol{R}$ will be referred to by $\boldsymbol{R}_+$. The symbol $\boldsymbol{N}$ stands for the set of positive integers. By K, S_n we denote a field, respectively the symmetric group of n elements.

Let $n \in \boldsymbol{N}$, $n \geq 3$, $\sigma \in S_n$, $\sigma = (1, 2, \ldots, n)$ the cycle of length n, $[n] := \{1, 2, \ldots, n\}$, $\sigma^t[i] := \{\sigma^t(1), \sigma^t(2), \ldots, \sigma^t(i)\}$ for any $1 \leq i \leq n-1$ and $\{e_i\}_{1 \leq i \leq n}$ be the canonical base of $\boldsymbol{R}^n$. For a vector $\alpha \in \boldsymbol{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we will denote $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

For the monomial $x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in K[x_1, \ldots, x_n]$ with $a = (a_1, \ldots, a_n) \in \boldsymbol{N}^n$ we set $\log(x^a) = a$. Given a set A of monomials, the *log set of A*, denoted log(*A*), consists of all $\log(x^a)$ with $x^a \in A$.

If $0 \neq a \in \boldsymbol{R}^n$ we will denote by $H_a^+ = \{x \in \boldsymbol{R}^n \mid < x, a > \geq 0\}$, where < , > is the usual inner product in $\boldsymbol{R}^n$. Recall that a *polyhedral cone* $Q \subset \boldsymbol{R}^n$ is the intersection of a finite number of closed subspaces of the form H_a^+. If a polyhedral cone Q is written as

$$Q = H_{a_1}^+ \cap ... \cap H_{a_r}^+$$

such that no $H_{a_i}^+$ can be omitted, then we say that this is an *irreducible representation* of Q.

If $A = \{\gamma_1, ..., \gamma_r\}$ is a finite set of points in $\boldsymbol{R}^n$ the *cone* generated by A, denoted by $\boldsymbol{R}_+ A$, is defined as $\boldsymbol{R}_+ A = \{\sum_{i=1}^{r} a_i \gamma_i \mid a_i \in \boldsymbol{R}_+ \textit{ for all } 1 \leq i \leq n\}$. An important fact is that Q is a polyhedral cone in $\boldsymbol{R}^n$ if and only if there exists a finite set $A \subset \boldsymbol{R}^n$ such that $\boldsymbol{Q} = \boldsymbol{R}_+ A$ (see [1]).

If A_i are some nonempty subsets of $[n]$ for $1 \leq i \leq m$, $m \geq 2$, $\mathcal{A} = \{\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{m}}\}$ then the set of the vectors $\sum_{k=1}^{m} e_{i_k}$ with $i_k \in A_k$ is the base of a polymatroid, called the *transversal polymatroid presented by* $\mathcal{A}$. The *base ring* of a transversal polymatroid presented by $\mathcal{A}$ is the ring:

$$K[\mathrm{A}] := K[x_{i_1} ... x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m].$$

We know by [3] that the K – algebra K[A] is normal and hence Cohen-Macaulay. From [5] we know that the transversal polymatroid presented by

$$\mathrm{A} = \{A_k \mid A_{\sigma^t(k)} = [n], \textit{ if } k \in [i] \cup \{n\} \textit{ and } A_{\sigma^t(k)} = [n] \setminus \sigma^t[i], \textit{ if } k \in [n-1] \setminus [i]\}$$

has the base ring associated K[A] a Gorenstein ring for any $1 \leq i \leq n-1$ and $0 \leq t \leq n-1$.

We put:

$$\nu_{\sigma^t[i]} = -(n-i-1) \sum_{k=1}^{i} e_{\sigma^t(k)} + (i+1) \sum_{k=i+1}^{n} e_{\sigma^t(k)}$$

for any $1 \leq i \leq n-1$ and $0 \leq t \leq n-1$.

The cone generated by the exponent vectors of the monomials defining the base ring $K[\mathcal{A}]$,

$$A := \{\log(x_{j_1} ... x_{j_n}) \mid j_k \in A_k, \, k \in [n]\} \subset \boldsymbol{N}^n,$$

has the irreducible representation:

$$\boldsymbol{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_{\sigma^t[i]}, \nu_{\sigma^k[n-1]} \mid 0 \leq k \leq n-1\}$.

It easy to see that for any $1 \leq i \leq n-1$ and $0 \leq t \leq n-1$

$$A = \{\alpha \in \boldsymbol{N}^n \mid |\alpha| = n, \, 0 \leq \alpha_{t+1} + ... + \alpha_{t+i} \leq i+1\}, \textit{ if } i + t \leq n$$

and

$$A=\{\alpha\in \mathbf{N}^n \mid |\alpha|=n,\ 0\le \sum_{s=1}^{i+t-n}\alpha_s + \sum_{s=t+1}^{n}\alpha_s \le i+1\},\ \textit{if}\ i+t>n.$$

Example.

Let $\mathcal{A}=\{A_1=\{1,2,3\}, A_2=\{1,2,3\}, A_3=\{1,3\}$, thus $i=1$ and $t=1$.

$A=\{(3,0,0),(2,0,1),(2,1,0),(1,1,1),(1,0,2),(1,2,0),(0,2,1),(0,1,2),(0,0,3)\}$.

$\mathbf{R}_+A=H^+_{a_1}\cap H^+_{a_2}\cap H^+_{a_3}\cap H^+_{a_4}$, where $a_1=(1,0,0)$, $a_2=(0,1,0)$, $a_3=(0,0,1)$, $a_4=(2,-1,2)$.

It is clearly that $A=\{\alpha\in \mathbf{N}^3 \mid |\alpha|=3,\ 0\le\alpha_2\le 2\}$.

Ehrhart Function

We consider a fixed set of distinct monomials $F=\{x^{\alpha_1},\dots,x^{\alpha_r}\}$ in a polynomial ring $R=K[x_1,\dots,x_n]$ over a field K. Let

$$\mathrm{P}=conv(\log(F))$$

be the convex hull of the set $\log(F)=\{\alpha_1,\dots,\alpha_r\}$.

The *normalized Ehrhart ring of* P is the graded algebra:

$$A_{\mathrm{P}}=\bigoplus_{j=1}^{\infty}(A_{\mathrm{P}})_j\subset R[T],$$

where the j^{th} component is given by

$$(A_P)_j=\sum_{\alpha\in \mathrm{Z}\log(\mathrm{F})\cap j\mathrm{P}} K\,x^{\alpha}T^j,$$

where $\mathbf{Z}(log(F))$ the group generated by $log(F)$ and $j\,\mathrm{P}=\{(ja_1,\dots,ja_n)\mid (a_1,\dots,a_n)\in P\}$.

The normalized Ehrhart function of P is defined as

$$E_{\mathrm{P}}(j)=\dim_K(A_{\mathrm{P}})_j=|\,\mathbf{Z}\log(F)\cap j\mathrm{P}\,|.$$

From [9. Proposition 7.2.39. and Corollary 7.2.45] we have the following important result.

Theorem 1. If $K[F]$ is a standard graded subalgebra of R and h is the Hilbert function of $K[F]$, then:

a) $h(j)\le E_P(j)$ for all $j\ge 0$, and
b) $h(j)=E_P(j)$ for all $j\ge 0$ if and only if $K[F]$ is normal.

The Computation of Hilbert Function

Let $n\ge 3$, $1\le i_1,i_2\le n-2$, $0\le t_2\le n-1$. We consider two transversal polymatroids presented by:

$$\mathcal{A}=\{A_k\mid A_k=[n],\ \textit{if}\ k\in[i_1]\cup\{n\},\ A_k=[n]\setminus[i_1],\ \textit{if}\ k\in[n-1]\setminus[i_1]\}$$

and

$$\mathcal{B} = \{B_k \mid B_{\sigma^{t_2}(k)} = [n], \textit{ if } k \in [i_2] \cup \{n\} \textit{ and } B_{\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2], \textit{ if } k \in [n-1] \setminus [i_2]\}$$

such that A, respectively B are the exponent vectors of the monomials defining the base rings associated to transversal polymatroid presented by $\mathcal{A}$, respectively $\mathcal{B}$. From [5] we know that the base rings $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$ are Gorenstein rings and the cones generated by the exponent vectors of the monomials defining the base ring associated to the transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$ are:

$$\mathbf{R}_+ A = \bigcap_{a \in N_1} H_a^+, \quad \mathbf{R}_+ B = \bigcap_{a \in N_2} H_a^+,$$

where

$$N_1 = \{\nu_{\sigma^0[i_1]}, \nu_{\sigma^k[n-1]} \mid 0 \le k \le n-1\},$$

$$N_2 = \{\nu_{\sigma^{t_2}[i_2]}, \nu_{\sigma^k[n-1]} \mid 0 \le k \le n-1\},$$

$$A = \{\log(x_{j_1} \ldots x_{j_n}) \mid j_k \in A_k, \ k \in [n]\} \subset \mathbf{N}^n$$

and

$$B = \{\log(x_{j_1} \ldots x_{j_n}) \mid j_k \in B_k, \ k \in [n]\} \subset \mathbf{N}^n.$$

It is to see that

$$A = \{\alpha \in \mathbf{N}^n \mid |\alpha| = n, \ 0 \le \alpha_1 + \ldots + \alpha_{i_1} \le i_1 + 1\}$$

and

$$B = \{\alpha \in \mathbf{N}^n \mid |\alpha| = n, \ 0 \le \alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \le i_2 + 1\}, \textit{ if } i_2 + t_2 \le n$$

or

$$B = \{\alpha \in \mathbf{N}^n \mid |\alpha| = n, \ 0 \le \sum_{s=1}^{i_2+t_2-n} \alpha_s + \sum_{s=t_2+1}^{n} \alpha_s \le i_2 + 1\}, \textit{ if } i_2 + t_2 > n.$$

From [6] we know that the K – algebra $K[A \cap B]$ is Gorenstein. Next, we present a necessary and sufficient conditions such that $K[A \cap B]$ is the base ring associated to some transversal polymatroid.

Theorem 2 ([6]). Let $n \ge 3$, $1 \le i_1, i_2 \le n-2$, $0 \le t_2 \le n-1$. We consider two transversal polymatroids presented by:

$$\mathrm{A} = \{A_k \mid A_k = [n], \textit{ if } k \in [i_1] \cup \{n\}, \ A_k = [n] \setminus [i_1], \textit{ if } k \in [n-1] \setminus [i_1]\}$$

and

$$\mathcal{B} = \{B_k \mid B_{\sigma^{t_2}(k)} = [n], \textit{ if } k \in [i_2] \cup \{n\} \textit{ and } B_{\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2], \textit{ if } k \in [n-1] \setminus [i_2]\}$$

such that A, respectively B is the exponent vectors of the monomials defining the base rings associated to transversal polymatroid presented by $\mathcal{A}$, respectively $\mathcal{B}$. Then, the K – algebra $K[A \cap B]$ is the base ring associated to a transversal polymatroid if and only if one of the following conditions hold:

a) $i_1 = 1$.

b) $i_1 \geq 2$ *and* $t_2 = 0$.

c) $i_1 \geq 2$ *and* $t_2 = i_1$.

d) $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$ *and* $i_2 \in \{1, \ldots, i_1 - t_2\} \cup \{n - t_2, \ldots, n-2\}$

e) $i_1 \geq 2$, $i_1 + 1 \leq t_2 \leq n-1$ *and* $i_2 \in \{1, \ldots, n - t_2\} \cup \{n - t_2 + i_1, \ldots, n-2\}$.

Next in this section we will compute the Hilbert function and the Hilbert series for the K – algebra $K[A \cap B]$, where A, respectively B satisfies the hypothesis of Theorem 2.

Proposition 1. The Hilbert function of the K – algebra $K[A \cap B]$, where A, respectively B satisfies the hypothesis of Theorem 2 is:

i) If $i_1 < i_2$ and $t_2 = 0$, then

$$h(t) = \sum_{k=0}^{t(i_1+1)} \sum_{r=0}^{t(i_2+1)-k} \binom{k + i_1 - 1}{k} \binom{r + i_2 - i_1 - 1}{r} \binom{nt - r - k + n - i_2 - 1}{nt - r - k}$$

for any $t \geq 0$.

ii) If $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$, $1 \leq i_2 \leq i_1 - t_2$ or $i_1 > i_2$ *and* $t_2 = 0$, then

$$h(t) = \sum_{k=0}^{t(i_2+1)} \sum_{r=0}^{t(i_1+1)-k} \binom{k + i_2 - 1}{k} \binom{r + i_1 - i_2 - 1}{r} \binom{nt - r - k + n - i_1 - 1}{nt - r - k},$$

for any $t \geq 0$.

iii) If $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$, $n - t_2 \leq i_2 \leq n-2$ or $t_2 = i_1$ *and* $i_1 + i_2 > n$, then

$$h(t) = \sum_{s=t(n-i_1-1)}^{t(i_2+1)} \sum_{r=t(n-i_2-1)}^{t(i_1+1)} \binom{nt - s - r + i_1 + i_2 - n - 1}{nt - s - r} \binom{s + n - i_1 - 1}{s} \binom{r + n - i_2 - 1}{r}$$

for any $t \geq 0$.

iv) If $i_1 \geq 2$, $i_1 + 1 \leq t_2 \leq n-1$, $1 \leq i_2 \leq n - t_2$ or $t_2 = i_1$ *and* $i_1 + i_2 < n$, then

$$h(t) = \sum_{k=0}^{t(i_1+1)} \sum_{r=0}^{t(i_2+1)} \binom{k + i_1 - 1}{k} \binom{r + i_2 - 1}{r} \binom{nt - k - r + n - i_1 - i_2 - 1}{nt - k - r}$$

for any $t \geq 0$.

v) If $i_1 \geq 2$, $i_1 + 1 \leq t_2 \leq n-1$, $n - t_2 + i_1 \leq i_2 \leq n-2$, then

$$h(t) = \sum_{k=0}^{t(i_1+1)} \sum_{r=0}^{t(i_2+1)-k} \binom{k + i_1 - 1}{k} \binom{r + i_2 - i_1 - 1}{r} \binom{nt - k - r + n - i_2 - 1}{nt - k - r}$$

for any $t \geq 0$.

vi) If $i_1 \geq 2$, $t_2 = i_1$, $i_1 + i_2 = n$, then

$$h(t) = \sum_{k=0}^{t(i_1+1)} \sum_{r=0}^{t(i_2+1)t} \binom{k + i_1 - 1}{k} \binom{r + i_2 - 1}{r}$$

for any $t \geq 0$.

Proof. From [3] we know that the K – algebra $K[A \cap B]$ is normal. Therefore, to compute the Hilbert function of $\mathcal{P}$, where $\mathcal{P} = conv(A \cap B)$ it is equivalently to compute Ehrhart function of $\mathcal{P}$. It is clearly enough to show that $\mathcal{P}$ is the intersection of the cone $\boldsymbol{R}_+(A \cap B)$ with the hyperplane $x_1 + \ldots + x_n = n$.

Now we will start to study all six cases.

i) If $i_1 < i_2$ and $t_2 = 0$, then

$$\mathcal{P} = \left\{ \begin{array}{l} \alpha \in \boldsymbol{R}^n \mid \alpha_k \geq 0 \, \forall \, k \in [n], \, |\alpha| = n, \; 0 \leq \alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \alpha_1 + \ldots + \alpha_{i_1} + \ldots + \alpha_{i_2} \leq i_2 + 1 \end{array} \right\},$$

whence it follows that

$$t\mathcal{P} = \left\{ \begin{array}{l} \alpha \in \boldsymbol{R}^n \mid \alpha_k \geq 0 \, \forall \, k \in [n], \, |\alpha| = nt, \; 0 \leq \alpha_1 + \ldots + \alpha_{i_1} \leq (i_1 + 1)t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \alpha_1 + \ldots + \alpha_{i_1} + \ldots + \alpha_{i_2} \leq (i_2 + 1)t \end{array} \right\}.$$

Since for any $0 \leq k \leq t(i_1 + 1)$ *and* $0 \leq r \leq t(i_2 + 1) - k$ the equations $\alpha_1 + \ldots + \alpha_{i_1} = k$ has $\binom{k + i_1 - 1}{k}$ nonnegative integer solutions, $\alpha_{i_1+1} + \ldots + \alpha_{i_2} = r$ has $\binom{r + i_2 - i_1 - 1}{r}$ nonnegative integer solutions and $\alpha_{i_2+1} + \ldots + \alpha_n = np - k - r$ has $\binom{nt - r - k + n - i_2 - 1}{nt - r - k}$ nonnegative integer solutions, we get that

$$h(t) = E_P(t) = |\boldsymbol{Z}\, A \cap \mathrm{tP}| = \sum_{k=0}^{t(i_1+1)} \sum_{r=0}^{t(i_2+1)-k} \binom{k + i_1 - 1}{k} \binom{r + i_2 - i_1 - 1}{r} \binom{nt - r - k + n - i_2 - 1}{nt - r - k}.$$

ii) If $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$, $1 \leq i_2 \leq i_1 - t_2$ or $i_1 > i_2$ *and* $t_2 = 0$, then

$$\mathcal{P} = \left\{ \begin{array}{l} \alpha \in \boldsymbol{R}^n \mid \alpha_k \geq 0 \, \forall \, k \in [n], \, |\alpha| = n, \; 0 \leq \alpha_1 + \ldots + \alpha_{i_1} \leq i_1 + 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq i_2 + 1 \end{array} \right\},$$

whence it follows that

$$t\mathcal{P} = \left\{ \begin{array}{l} \alpha \in \boldsymbol{R}^n \mid \alpha_k \geq 0 \, \forall \, k \in [n], \, |\alpha| = nt, \; 0 \leq \alpha_1 + \ldots + \alpha_{i_1} \leq (i_1 + 1)t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} \leq (i_2 + 1)t \end{array} \right\},$$

Since for any

$$0 \leq k \leq t(i_2 + 1) \; \text{ and } \; 0 \leq r \leq t(i_1 + 1) - k\,,$$

the equations $\alpha_{t_2+1} + \ldots + \alpha_{t_2+i_2} = k$ has $\binom{k + i_2 - 1}{k}$ nonnegative integer solutions,

$\alpha_1+...+\alpha_{t_2}+\alpha_{t_2+i_2+1}+...+\alpha_{i_1}=r$ has $\begin{pmatrix} r+i_1-i_2-1 \\ r \end{pmatrix}$ nonnegative integer solutions and $\alpha_{i_1+1}+...+\alpha_n=np-k-r$ has $\begin{pmatrix} nt-r-k+n-i_1-1 \\ nt-r-k \end{pmatrix}$ nonnegative integer solutions, we get that

$$h(t)=E_P(t)=\left|\mathbf{Z}\,A\cap \mathrm{tP}\right|=\sum_{k=0}^{t(i_2+1)}\sum_{r=0}^{t(i_1+1)-k}\begin{pmatrix} k+i_2-1 \\ k \end{pmatrix}\begin{pmatrix} r+i_1-i_2-1 \\ r \end{pmatrix}\begin{pmatrix} nt-r-k+n-i_1-1 \\ nt-r-k \end{pmatrix}.$$

It is easy to see that for $i_1>i_2$ *and* $t_2=0,$ we have the same formula for Hilbert function.

iii) If $i_1\geq 2,\ 1\leq t_2\leq i_1-1,\ n-t_2\leq i_2\leq n-2$ or $t_2=i_1$ *and* $i_1+i_2>n,$ then

$$\mathcal{P}=\left\{\begin{matrix}\alpha\in \boldsymbol{R}^{\boldsymbol{n}}\,|\,\alpha_k\geq 0\,\forall\, k\in[n],\ |\alpha|=n,\ 0\leq \alpha_1+...+\alpha_{i_2+t_2-n}+\alpha_{t_2+1}+...+\alpha_n\leq i_2+1, \\ 0\leq \alpha_1+...+\alpha_{i_1}\leq i_1+1\end{matrix}\right\},$$

whence it follows that

$$t\mathcal{P}=\left\{\begin{matrix}\alpha\in \boldsymbol{R}^{\boldsymbol{n}}\,|\,\alpha_k\geq 0\,\forall\, k\in[n],\ |\alpha|=nt,\ 0\leq \alpha_1+...+\alpha_{i_2+t_2-n}+\alpha_{t_2+1}+...+\alpha_n\leq (i_2+1)t, \\ 0\leq \alpha_1+...+\alpha_{i_1}\leq (i_1+1)t\end{matrix}\right\},$$

Since for any

$$t(n-i_1-1)\leq s\leq t(i_2+1)\ \text{and}\ t(n-i_2-1)\leq r\leq t(i_1+1)$$

the equations $\alpha_{i_1+1}+...+\alpha_n=s$ has $\begin{pmatrix} s+n-i_1-1 \\ s \end{pmatrix}$ nonnegative integer solutions,

$\alpha_{t_2+i_2-n+1}+...+\alpha_{t_2}=r$ has $\begin{pmatrix} r+n-i_2-1 \\ r \end{pmatrix}$ nonnegative integer solutions and

$\alpha_1+...+\alpha_{i_2+t_2-n}+\alpha_{t_2+1}+...+\alpha_{i_1}=nt-s-r$ has $\begin{pmatrix} nt-s-r+i_1+i_2-n-1 \\ nt-s-r \end{pmatrix}$ nonnegative integer solutions, we get that

$$h(t)=E_P(t)=\left|\mathbf{Z}\,A\cap \mathrm{tP}\right|=$$

$$=\sum_{s=t(n-i_1-1)}^{t(i_2+1)}\sum_{r=t(n-i_2-1)}^{t(i_1+1)}\begin{pmatrix} nt-s-r+i_1+i_2-n-1 \\ nt-s-r \end{pmatrix}\begin{pmatrix} s+n-i_1-1 \\ s \end{pmatrix}\begin{pmatrix} r+n-i_2-1 \\ r \end{pmatrix}.$$

It is easy to see that for $t_2=i_1$ *and* $i_1+i_2>n$ we have the same formula for Hilbert function.

iv) If $i_1\geq 2,\ i_1+1\leq t_2\leq n-1,\ 1\leq i_2\leq n-t_2$ or $t_2=i_1$ *and* $i_1+i_2<n$, then

$$\mathcal{P}=\left\{\begin{matrix}\alpha\in \boldsymbol{R}^{\boldsymbol{n}}\,|\,\alpha_k\geq 0\,\forall\, k\in[n],\ |\alpha|=n,\ 0\leq \alpha_1+...+\alpha_{i_1}\leq i_1+1, \\ 0\leq \alpha_{t_2+1}+...+\alpha_{t_2+i_2}\leq i_2+1\end{matrix}\right\},$$

whence it follows that

$$t\mathcal{P}=\left\{\begin{array}{r}\alpha\in\boldsymbol{R}^{n}\,|\,\alpha_k\geq 0\,\forall\,k\in[n],\,|\alpha|=nt,\ 0\leq\alpha_1+\ldots+\alpha_{i_1}\leq(i_1+1)t,\\ 0\leq\alpha_{t_2+1}+\ldots+\alpha_{t_2+i_2}\leq(i_2+1)t\end{array}\right\}.$$

Since for any $0\leq k\leq t(i_1+1)$ and $0\leq r\leq t(i_2+1)$ the equations $\alpha_1+\ldots+\alpha_{i_1}=k$ has $\binom{k+i_1-1}{k}$ nonnegative integer solutions , $\alpha_{t_2+1}+\ldots+\alpha_{t_2+i_2}=r$ has $\binom{r+i_2-1}{r}$ nonnegative integer solutions and $\alpha_{i_1+1}+\ldots+\alpha_{t_2}+\alpha_{t_2+i_2+1}+\ldots+\alpha_n=np-k-r$ has $\binom{nt-r-k+n-i_1-i_2-1}{nt-r-k}$ nonnegative integer solutions, we get that

$$h(t)=E_P(t)=|\boldsymbol{Z}\,A\cap\mathrm{tP}|=\sum_{k=0}^{t(i_1+1)}\sum_{r=0}^{t(i_2+1)}\binom{k+i_1-1}{k}\binom{r+i_2-1}{r}\binom{nt-k-r+n-i_1-i_2-1}{nt-k-r}.$$

It is easy to see that for $t_2=i_1$ and $i_1+i_2<n$ we have the same formula for Hilbert function.

v) If $i_1\geq 2,\ i_1+1\leq t_2\leq n-1,\ n-t_2+i_1\leq i_2\leq n-2,$ then

$$\mathcal{P}=\left\{\begin{array}{r}\alpha\in\boldsymbol{R}^{n}\,|\,\alpha_k\geq 0\,\forall\,k\in[n],\,|\alpha|=n,\ 0\leq\alpha_1+\ldots+\alpha_{i_1}\leq i_1+1,\\ 0\leq\alpha_1+\ldots+\alpha_{i_1}+\ldots+\alpha_{i_2+t_2-n}+\alpha_{t_2+1}+\ldots+\alpha_n\leq i_2+1\end{array}\right\},$$

whence it follows that

$$t\mathcal{P}=\left\{\begin{array}{r}\alpha\in\boldsymbol{R}^{n}\,|\,\alpha_k\geq 0\,\forall\,k\in[n],\,|\alpha|=nt,\ 0\leq\alpha_1+\ldots+\alpha_{i_1}\leq(i_1+1)t,\\ 0\leq\alpha_1+\ldots+\alpha_{i_1}+\ldots+\alpha_{i_2+t_2-n}+\alpha_{t_2+1}+\ldots+\alpha_n\leq(i_2+1)t\end{array}\right\}.$$

Since for any $0\leq k\leq t(i_1+1)$ and $0\leq r\leq t(i_2+1)-k$ the equations $\alpha_1+\ldots+\alpha_{i_1}=k$ has $\binom{k+i_1-1}{k}$ nonnegative integer solutions, $\alpha_{i_1+1}+\ldots+\alpha_{i_2+t_2-n}+\alpha_{t_2+1}+\ldots+\alpha_n=r$ has $\binom{r+i_2-i_1-1}{r}$ nonnegative integer solutions and $\alpha_{i_2+t_2-n+1}+\ldots+\alpha_{t_2}=np-k-r$ has $\binom{nt-r-k+n-i_2-1}{nt-r-k}$ nonnegative integer solutions, we get that

$$h(t)=E_P(t)=|\boldsymbol{Z}\,A\cap\mathrm{tP}|=\sum_{k=0}^{t(i_1+1)}\sum_{r=0}^{t(i_2+1)-k}\binom{k+i_1-1}{k}\binom{r+i_2-i_1-1}{r}\binom{nt-k-r+n-i_2-1}{nt-k-r}.$$

vi) If $i_1\geq 2,\ t_2=i_1,\ i_1+i_2=n$, then

$$\mathcal{P}=\left\{\begin{array}{r}\alpha\in\boldsymbol{R}^{n}\,|\,\alpha_k\geq 0\,\forall\,k\in[n],\,|\alpha|=n,\ 0\leq\alpha_1+\ldots+\alpha_{i_1}\leq i_1+1,\\ 0\leq\alpha_{i_1+1}+\ldots+\alpha_n\leq i_2+1\end{array}\right\},$$

whence it follows that

$$t\mathcal{P} = \left\{ \begin{array}{l} \alpha \in \boldsymbol{R}^n \mid \alpha_k \geq 0 \,\forall\, k \in [n],\ |\alpha| = nt,\ 0 \leq \alpha_1 + \ldots + \alpha_{i_1} \leq (i_1+1)t, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \alpha_{i_1+1} + \ldots + \alpha_n \leq (i_2+1)t \end{array} \right\},$$

Since for any $0 \leq k \leq t(i_1+1)$ *and* $0 \leq r \leq t(i_2+1)$ the equations $\alpha_1 + \ldots + \alpha_{i_1} = k$ has $\binom{k+i_1-1}{k}$ nonnegative integer solutions, $\alpha_{i_1+1} + \ldots + \alpha_n = r$ has $\binom{r+i_2-1}{r}$ nonnegative integer solutions, we get that

$$h(t) = E_P(t) = |\mathbf{Z}\, A \cap \mathrm{tP}| = \sum_{k=0}^{t(i_1+1)} \sum_{r=0}^{t(i_2+1)t} \binom{k+i_1-1}{k}\binom{r+i_2-1}{r}.$$

The Computation of Hilbert Series

Since the *a*- invariant of $K[A \cap B]$ is $a(K[A \cap B]) = -1$ (see [6]), it follows that to compute the Hilbert series of $K[A \cap B]$ it is necesary to know the first *n* values of the Hilbert function of $K[A \cap B]$. From [6] the Hilbert series of $K[A \cap B]$ is

$$H_{K[A \cap B]}(t) = \frac{1 + h_1 t + \ldots + h_{n-1} t^{n-1}}{(1-t)^n},$$

where

$$n = \dim(K[A \cap B]),\ h_j = \sum_{s=0}^{j} (-1)^s h(j-s) \binom{n}{s}$$

and $h(s)$ is the Hilbert function of $K[A \cap B]$.

References

1. Bruns, W., Herzog, J. - *Cohen-Macaulay rings*, Revised Edition, Cambridge, 1997
2. Bruns, W., Ichim, B. – Normaliz, 2.1, available from: http://www.mathematik.uni-osnabrueck.de/normaliz/index.html, accessed on February 06, 2009
3. Herzog, J., Hibi, T. - Discrete Polymatroids, *Journal of Algebraic Combinatorics*, 16(2002), no. 3, pp. 239-268, 2002
4. Herzog, J., Hibi, T., Vlădoiu, M. - Ideals of fiber type and polymatroids, *Osaka Journal of Mathematics*, 42(2005), pp. 807-829, 2005
5. Ştefan, A. - A class of transversal polymatroids with Gorenstein base ring, *Bulletin Mathematique de la Société des Sciences Mathématiques de Roumanie*, Tome 51 (99) no. 1, 2008, pp. 68-79, 2008
6. Ştefan, A. - Intersections of base rings associated to transversal polymatroids, *Bulletin Mathematique* de *la Société des Sciences Mathématiques de Roumanie*, Tome 52 (100) no. 1, 2009, pp. 66-78, 2009
7. Ştefan, A. - The base ring associated to a transversal polymatroids, *Analele Universităţii Bucureşti*, Anul LVIII, Nr. 1, 2009, 31-67
8. Ştefan, A. - The type of the base ring associated to a transversal polymatroid, *Contemporary Mathematics, "Combinatorial Aspects of Commutative Algebra",* vol. 502, pp. 168-184, 2009
9. Villarreal, R. - *Monomial Algebras*, Marcel Dekker, New-York, 2001
10. Villarreal, R. - Rees cones and monomial rings of matroids, *Linear Algebra and its Applications*, 428(2008), pp. 2933-2940, 2008

11. V l ă d o i u , M . - Discrete Polymatroids, *Analele Ştiintifice ale Universitatii Ovidius Constanța*, *Seria Matematica*, 14(2006), pp. 29-112, 2006

Funcția Hilbert a intersecției a două inele bază asociate polimatroizilor transversali

Rezumat

Intersecția a două inele bază asociate unor polimatroizi transversali nu este in general inel bază asociat unui polimatroid transversal. În această lucrare, determinăm funcția Hilbert şi seria Hilbert în cazul în care intersecția a două inele bază este un inel bază asociat unui polimatroid transversal.