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Common Fixed Points for (d)-Contractive Multifunctions in Metric Spaces

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Abstract

A common fixed point theorem is given in complete metric spaces (X,d) for pairs of multifunctions constrained by a contractive condition only using metric d, without using the Hausdorff metric.

Key words: *multifunctions, (d)-contractive multifunctions, fixed point*

Introduction

Let (X,d) be a metric space, $P(X)$ the family of non empty subsets of X and $P_{clb}(X)$ the closed non empty and bounded subsets from X.

Let be the Hausdorff metric

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

with $A,B \in P_{clb}(X)$ and $d(a,B) = \inf_{b\in B} d(a,b)$.

For mappings of the type $f: X \to P(X)$, called multifunctions, there are many fixed point theorems which contain a contractive condition using Hausdorff metric. In paper [1] fixed point theorems are given for multifunctions which do not use the Hausdorff metric.

Common fixed point theorems for two or more multifunctions, using the Hausdorff metric, are given in [2], [3] and [4].

In this paper we will give a common fixed point theorem for multifunctions, without using the Hausdorff metric.

Pairs of (d)-Contractive Multifunctions

Definition. Let (X,d) be a metric space. We will say that $f,g: X \to P(X)$ form a pair of (d)-contractive multifunctions if there exists $q \in (0,1)$ such that for any $x \in X$ and for any $y \in f(x)(g(x))$ there exists $z \in g(y)(f(y))$ with the property

$$d(y,z) \le q d(x,y).$$

Let (X,d) be a metric space, $A,B\in P_{clb}(X)$ and $\alpha>1$. Particularizing lemma 1 (V) from [3] with the following enunciation:

for any $a\in A$, there exists $b\in B$ such that $d(a,b)\le \alpha H(A,B)$,

we obtain

Lemma 1. Let (X,d) be a metric space, $k>1$ and the multifunctions $f,g:X\to P_{clb}(X)$. Then, for any $x\in X$ and any $y\in f(x)$ (respectively $y\in g(x)$) there exists $z\in g(y)$ (respectively $z\in f(y)$) with the property

$$d(y,z)\le kH\left(f(x),g(y)\right),$$

respectively

$$d(y,z)\le kH\left(g(x),f(y)\right).$$

Lemma 2. Let (X,d) be a metric space. If the multifunctions $f,g:X\to P(X)$ verify the property: there exists $a,b,c\in \mathbf{R}_+$ with $a+b+c<1$ such that

$$H\left(f(x),g(y)\right)\le ad(x,y)+bd(x,f(x))+cd(y,g(y)),\ (\forall)x,y\in X$$

and respectively

$$H\left(g(x),f(y)\right)\le ad(x,y)+bd(x,g(x))+cd(y,f(y)),\ (\forall)x,y\in X,$$

then f, g form a pair of (d)-contractive multifunctions.

Proof. Particularly, the relations from this lemma take place for any $x\in X$ and for any $y\in f(x)$ (respectively $y\in g(x)$), while based on lemma 1, for $k>1$, we deduce that there exists $z\in g(y)$ (respectively $z\in f(y)$) such that

$$\begin{aligned} d(y,z)&\le kH\left(f(x),g(y)\right)\le\\ &\le k\left[ad(x,y)+bd(x,f(x))+cd(y,g(y))\right]\le\\ &\le k\left[ad(x,y)+bd(x,y)+cd(y,z)\right]\le\\ &\le k(a+b)d(x,y)+cd(y,z)\end{aligned}$$

and since $1-kc>0$, we obtain

$$d(y,z)\le k\frac{a+b}{1-kc}d(x,y).$$

Considering $1<k<\dfrac{1}{a+b+c}$, it results $q=k\dfrac{a+b}{1-kc}<1$ and the previous inequality becomes

$$d(y,z)\le qd(x,y),$$

which shows that f, g form a pair of (d)-contractive multifunctions.

Let (X,d) be a metric space and $f:X\to P(X)$ a multifunction. The following property will be used later in this paper:

$$(i)\begin{cases}\text{for any convergent sequence } (x_n)_{n\ge 0} \text{ from } X \text{ with}\\ x_{n+1}\in f(x_n), n\ge 0 \text{ and } \lim\limits_{n\to\infty} x_n = x, \text{ it results } x\in f(x).\end{cases}$$

We mention that superior semicontinous multifunctions have this property.

Common Fixed Points for (d)-Contractive Multifunctions

Theorem. If (X,d) is a complete metric space, then any pair of (d)-contractive multifunctions $f,g: X \to P(X)$ which verify property (i) has a common fixed point, that is there exists $w \in X$ such that

$$w \in f(w),\ w \in g(w).$$

Proof. Let $x_0 \in X$ be arbitrary fixed. If $x_0 \in f(x_0) \bigcap g(x_0)$, then x_0 is a common fixed point and the theorem is proven. If $x_0 \notin f(x_0) \bigcap g(x_0)$, then there exists $x_1 \in f(x_0) \bigcup g(x_0)$, with $x_1 \neq x_0$. Let $x_1 \in f(x_0)$. Then there exists $x_2 \in g(x_1)$ with $x_2 \neq x_1$ and

$$d(x_1, x_2) \leq q d(x_0, x_1).$$

Continuing with this procedure, we obtain a sequence $(x_n)_{n\geq 0}$ with

$$x_{2n+1} \in f(x_{2n}),\ x_{2n+2} \in g(x_{2n+1}),\ (\forall) n \geq 0,$$

which verify the condition

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1),\ (\forall) n \geq 0,$$

from where we deduce

$$d(x_n, x_{n+p}) \leq \frac{q^n}{1-q} d(x_0, x_1),\ (\forall) n, p \in \mathbf{N}.$$

This shows that $(x_n)_{n\geq 0}$ is a Cauchy sequence and therefore convergent, since the space (X,d) is presumed complete. Let $w = \lim\limits_{n\to\infty} x_n$. The multifunctions f, g verify property (i) and consequently it results

$$w \in f(w),\ w \in g(w),$$

that is w is common fixed point for the pair f, g and the theorem is completely proven.

Corollary 1. Let (X,d) be a complete metric space and $f,g: X \to P(X)$ two multifunctions which verify condition (i) and have the property that there exists $a,b \in \mathbf{R}_+$ with $a + 2b < 1$ such that for any $x \in X$, any $y \in f(x)\ (g(x))$, there exists $z \in g(y)\ (f(y))$ with $z \neq y$ and

$$d(y,z) \leq a d(x,y) + b d(x,z).$$

Then f, g have a common fixed point.

Proof. The condition from above gives

$$d(y,z) \leq a d(x,y) + b\left[d(x,y) + d(y,z)\right],$$

that is

$$d(y,z) \leq \frac{a+b}{1-b} d(x,y)$$

and as $\frac{a+b}{1-b} < 1$, we apply the theorem.

Corollary 2. Let (X,d) be a complete metric space and $f,g: X \to P_{clb}(X)$ multifunctions with the property from lemma 2. Then f, g have a common fixed point.

Proof. We will show that the pair f , g satisfy the theorem's conditions. Based on lemma 2, it results that f , g form a pair of (d)-contractive multifunctions. From the proof of the theorem we deduce that for any $x_0 \in X$, there exists a convergent sequence $(x_n)_{n\geq 0}$ from X , such that

$$x_{2n+1} \in f(x_{2n}),\ x_{2n+2} \in g(x_{2n+1}),\ (\forall) n \geq 0 .$$

Lets show that the multifunctions f , g satisfy condition (i) . Let $w = \lim_{n\to\infty} x_n$. We sequentially have

$$\begin{aligned} d(w, f(w)) &\leq d(w, x_{2n+2}) + d(x_{2n+2}, f(w)) \leq \\ &\leq d(w, x_{2n+2}) + H\left(f(w), g(x_{2n+1})\right) \leq \\ &\leq d(w, x_{2n+2}) + ad(w, x_{2n+1}) + bd(w, f(w)) + cd(x_{2n+1}, g(x_{2n+1})) \leq \\ &\leq d(w, x_{2n+2}) + ad(w, x_{2n+1}) + bd(w, f(w)) + cd(x_{2n+1}, x_{2n+2}), \end{aligned}$$

that is

$$(1-b)d(w, f(w)) \leq d(w, x_{2n+2}) + ad(w, x_{2n+1}) + cd(x_{2n+1}, x_{2n+2}) ,$$

from where, for $n \to \infty$, it results $(1-c)d(w, f(w)) \leq 0$ or more

$$\inf_{t\in f(w)} d(w,t) = 0$$

and since $f(w)$ is a closed set, we deduce that $w \in f(w)$. Analogously we show that $w \in g(w)$. So f , g verify (i) and consequently we apply the theorem.

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Puncte fixe comune pentru multifuncții (d)-contractive în spații metrice

Rezumat

Se dă o teoremă de punct fix comun în spații metrice complete (X,d) *pentru perechi de multifuncții, supuse unei condiții de tip contracție, exprimată numai cu metrica d, fără a utiliza metrica Hausdorff.*