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Linear and Quadratic Interpolation of the Functions with Two Variable Values with Simple Nodes

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Abstract

The theme of interpolation of a function with two variable values is a complex matter which implies major difficulties when solving it concerning its form, the grade of the polynomial interpolation and the number of points from the definition field.The paper presents two cases of linear and quadratic interpolation which may be applied when solying problemas with elementary fields such as triangle and rectangle.

Key words: *interpolation, network, division, divided differences*

Introduction

Considering a function $f:[a,b]\rightarrow \mathbf{R}$ and a division of the interval $[a,b]$

$$\Delta:(x_1 = a < x_2 < x_3 < ... < x_n = b)$$

for which it is known the point $A_k(x_k, y_k)$ with $y_k = f(x_k)$ for $k \in \{1,2,...,n\}$.

It is asked to be determined a polynomial of the degree n named P which should approximate the function f on the interval $[a,b]$ so as:

$$P(x_k) = y_k = f(x_k) \text{ for } k \in \{1,2,...,n\}. \tag{1}$$

The answer to this problem is given by the formula of Lagrange [3]:

$$P(x) = \sum_{k=1}^{n} y_k \frac{Q_k(x)}{Q_k(x_k)}, \tag{2}$$

where

$$Q_k(x) = \frac{\omega(x)}{x - x_k} \text{ and } \omega(x) = \prod_{i=1}^{n}(x - x_i) \tag{3}$$

or by the formula of Newton [3] using the divided differences:

$$P(x) = P(f; x_1, x_2, ..., x_n; x) = f[x_1] + f[x_1, x_2](x - x_1) + f[x_1, x_2, x_3]\cdot \\ \cdot (x - x_1)(x - x_2) + ... + f[x_1, x_2, ..., x_n](x - x_1)(x - x_2)...(x - x_{n-1}) \tag{4}$$

where the divided differences are:

$$f[x_1]=f(x_1)\ and\quad f[x_1,x_2,...,x_k]=\frac{f[x_1,...,x_{k-1}]-f[x_2,...,x_k]}{x_1-x_k} \tag{5}$$

or

$$f[x_1,x_2,...,x_k]=\int_0^1 dt_1\left(\int_0^{t_1} dt_2\left(...\int_0^{t_{k-2}} f^{(k-1)}(x_1+t_1(x_2-x_1)+...+t_{k-1}(x_k-x_{k-1}))dt_{k-1}...\right)\right) \tag{6}$$

Content

Considering the field $D=[a,b]\times[c,d]$ with the divisions

$$\Delta_1 : x_0=a<x_1<x_2<...<x_i<x_{i+1}<...<x_n=b$$

$$\Delta_2 : y_0=c<y_1<y_2<...<y_j<y_{j+1}<...<y_m=d\ ,$$

which determine the division: $\Delta=\Delta_1\times\Delta_2$ for the field D and the function $f:D\to IR$ for which it is known $f(x_i,y_j)\overset{not}{=}z_{i,j}$ with $i\in\{0,1,...,n\},\ j\in\{0,1,...,m\}$.

The notions of the network Δ points notes $N(x_i,y_i)\overset{notation}{=}N_{i,j}$ for $i\in\{0,1,...,n\}$ and $j\in\{0,1,...,m\}$.

A. Being determined a two grade polynomial $P_{i,j}$ in x and y which has to aproximate the function f on the field $D_{i,j}$, where

$$D_{i,j}=\{(x,y)|\ x_i\le x<x_{i+1}\quad \text{and}\ y_j\le y<y_{j+1}\} \tag{7}$$

and has to carry out the conditions:

$$\begin{cases} P_{i,j}(x_i,y_j)=f(x_i,y_j) \\ P_{i,j}(x_{i+1},y_j)=f(x_{i+1},y_{j+1}) \\ P_{i,j}(x_{i+1},y_{j+1})=f(x_{i+1},y_{j+1}) \\ P_{i,j}(x_i,y_{j+1})=f(x_i,y_{j+1}) \end{cases} \tag{8}$$

for $i\in\{0,1,...,n-1\}$ end $j\in\{0,1,...,m-1\}$. If the polynomial $P_{i,j}$ has the form:

$$P_{i,j}(x,y)=\alpha\cdot(x-x_i)^2+\beta(x-x_i)(y-y_j)+\gamma(y-y_j)^2+\delta \tag{9}$$

then imposing the conditions (8) we obtain:

$$\begin{cases} P_{i,j}(x_i,y_j)=\delta=f(x_i,y_j); \\ P_{i,j}(x_{i+1},y_j)=\alpha(x_{i+1}-x_i)^2+\delta=f(x_{i+1},y_j); \\ P_{i,j}(x_i,y_{j+1})=\gamma(y_{j+1}-y_j)^2+\delta=f(x_i,y_{j+1}); \\ P_{i,j}(x_{i+1},y_{j+1})=\alpha(x_{i+1}-x_i)^2+\beta(x_{i+1}-x_i)(y_{j+1}-y_j)+\gamma(y_{j+1}-y_j)^2+\delta= \\ =f(x_{i+1},y_{j+1}). \end{cases} \tag{10}$$

Solving system (10) and using the notations from [4] it is obtained:

$$\begin{cases} \delta = f(x_i, y_j); \\ \alpha = \dfrac{f(x_{i+1}, y_j) - f(x_i, y_j)}{(x_{i+1} - x_i)^2} = \dfrac{f[x_i, x_{i+1}; y_j]}{x_{i+1} - x_i}; \\ \gamma = \dfrac{f(x_i, y_{j+1}) - f(x_i, y_j)}{(y_{j+1} - y_j)^2} = \dfrac{f[x_i; y_j, y_{j+1}]}{y_{j+1} - y_j}; \\ \beta = \dfrac{f[x_{i+1}; y_j, y_{j+1}]}{x_{i+1} - x_i} - \dfrac{f[x_i; y_j, y_{j+1}]}{x_{i+1} - x_i} = f[x_i, x_{i+1}; y_j, y_j] \end{cases} . \quad (11)$$

Replacing it in relation (9) it is obtained:

$$P_{i,j}(x,y) = \frac{f[x_i, x_{i+1}; y_j]}{x_{i+1} - x_i}(x - x_i)^2 + f[x_i, x_{i+1}; y_j, y_{j+1}] \cdot (x - x_i)(y - y_j) + $$
$$+ \frac{f[x_i; y_j, y_{j+1}]}{y_{j+1} - y_j}(y - y_j)^2 + f(x_i, y_j) \quad (12)$$

for $(x, y) \in D_{i,j}$ and represents the two grade interpolation polynomial of the function f on the field $D_{i,j}$.

The polynomial basis $\{P_{i,j}\}$ with $i \in \{0,1,...,n-1\}$ and $j \in \{0,1,...,m-1\}$ interpolates the function f on the field $D = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{m-1} D_{i,j}$. In the particular case when the divisions Δ_1 and Δ_2 are equidistant, then $x_{i+1} - x_i = \frac{b-a}{n} = h$, $y_{j+1} - y_j = k = \frac{d-c}{m}$, and the polynom $P_{i,j}(x,y)$ given by (12) becomes:

$$P_{i,j}(x,y) = \frac{f[x_i, x_{i+1}; y_j]}{h}(x - x_i)^2 + f[x_i, x_{i+1}; y_j, y_{j+1}] \cdot (x - x_i)(y - y_j) + $$
$$+ \frac{f[x_i; y_j, y_{j+1}]}{k} \cdot (y - y_j)^2 + f(x_i, y_j) \text{ for } (x,y) \in D_{i,j}. \quad (13)$$

The error which is usually made when function f is replaced by the polynomial $P_{i,j}$ given (13) is[4]:

$$e_\tau = \left| f(x,y) - P_{i,j}(x,y) \right| \le M \cdot m_1 \cdot m_2, \quad (14)$$

where:

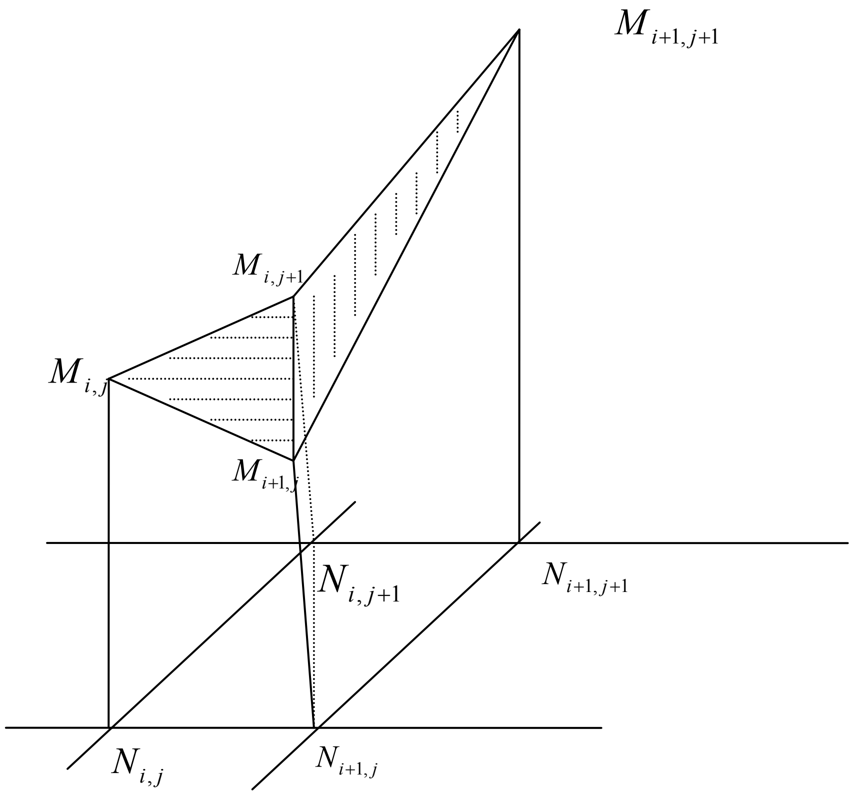
$$\begin{cases} M = \max\limits_{(x,y) \in D} \left| \dfrac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} \right|; \\ m_1 = \max\limits_{x \in [x_i, x_{i+1}]} \left| (x - x_i)(x - x_{i+1}) \right|; \\ m_2 = \max\limits_{y \in [y_j, y_{j+1}]} \left| (y - y_j)(y - y_{j+1}) \right|. \end{cases} \quad (15)$$

B. Considering the same field D with the division Δ and the subfields

$$D_{i,j} \ldots i \in \{0,1,2,\ldots,n-1\},\quad j \in \{0,1,\ldots,m-1\}.$$

For $D_{i,j}$ considering the triangular subfields T_1 and T_2:

$$\begin{cases} T_1 \text{ determined by the nodes } N(x_i, y_j), N(x_{i+1}, y_j) \text{ and } N(x_i, y_{j+1}) \\ T_2 \text{ determined by the nodes } N(x_{i+1}, y_j), N(x_{i+1}, y_{j+1}) \text{ and } N(x_i, y_{j+1}) \end{cases}.$$



The network nodes $N_{i,j}(x_i, y_j)$ determines in space the points $M_{i,j}(x_i, y_j, z_{i,j}) \overset{not}{=} M_{i,j}$

with $z_{i,j} = f(x_i, y_j)$, $i \in \{0,1,2,\ldots,n\}$ and $j \in \{0,1,2,\ldots,m\}$.

The points $M_{i,j}, M_{i+1,j}, M_{i,j+1}$ determines a plan noted $\alpha = \prod_{i,i+1,i}^{j,j,j+1}$ corresponding to the triangle T_1, which has the following equation:

$$(\alpha): \begin{vmatrix} x & y & z & 1 \\ x_i & y_j & z_{i,j} & 1 \\ x_{i+1} & y_j & z_{i+1,j} & 1 \\ x_i & y_{j+1} & z_{i,j+1} & 1 \end{vmatrix} = 0 \tag{16}$$

or

$$(\alpha\ ):\ \frac{z_{i+1,j}-z_{i,j}}{x_{i+1}-x_i}\left(x-x_i\right)+\frac{z_{i,j+1}-z_{i,j}}{y_{j+1}-y_j}\left(y-y_j\right)-\left(z-z_{i,j}\right)=0\,. \qquad (17)$$

Analogous, the points $M_{i+1,j}$, $M_{i+1,j+1}$, $M_{i,j+1}$ determines another noted plan: $\beta=\prod_{i+1,i+1,i}^{j,j+1,j+1}$ corresponding to the triangle T_2 which has the equation:

$$(\beta):\begin{vmatrix} x & y & z & 1 \\ x_{i+1} & y_j & z_{i+1,j} & 1 \\ x_{i+1} & y_{j+1} & z_{i+1,j+1} & 1 \\ x_i & y_{j+1} & z_{i,j+1} & 1 \end{vmatrix}=0 \qquad (18)$$

or

$$(\beta):\frac{z_{i+1,j+1}-z_{j,j+1}}{x_{i+1}-x_i}\left(x-x_{i+1}\right)+\frac{z_{i+1,j}-z_{i+1,j+1}}{y_{j+1}-y_j}\left(y-y_{j+1}\right)-\left(z-z_{i+1,j+1}\right)=0\,. \qquad (19)$$

From relation (17) results:

$$P_{i,j}(x,y)=z=z_{i,j}+\frac{z_{i+1,j}-z_{i,j}}{x_{i+1}-x_i}\left(x-x_i\right)+\frac{z_{i,j+1}-z_{i,j}}{y_{j+1}-y_j}\left(y-y_j\right)\ \text{ for } (\forall)(x,y)\in T_1 \qquad (20)$$

and from(19) it is obtained in an analogous manner:

$$P_{i,j}(x,y)=z=z_{i+1,j+1}+\frac{z_{i+1,j+1}-z_{i,j+1}}{x_{i+1}-x_i}\left(x-x_{i+1}\right)+\frac{z_{i+1,j+1}-z_{i+1,j}}{y_{j+1}-y_j}\left(y-y_{j+1}\right) \qquad (21)$$

for $(\forall)(x,y)\in T_2$.

The relations (20) and (21) aproximate the function f on the fields T_1 and T_2, with a grade one polynomial.

So, the linear interpolation polynomial corresponding to the field $D_{i,j}$ is :

$$P_{i,j}(x,y)=\begin{cases} z_{i,j}+\dfrac{z_{i+1,j}-z_{i,j}}{x_{i+1}-x_i}\left(x-x_i\right)+\dfrac{z_{i,j+1}-z_{i,j}}{y_{j+1}-y_j}\left(y-y_j\right)\ \text{ for } (x,y)\in T_1 \\ z_{i+1,j+1}+\dfrac{z_{i+1,j+1}-z_{i,j+1}}{x_{i+1}-x_i}\left(x-x_{i,j}\right)+\dfrac{z_{i+1,j+1}-z_{i+1,j}}{y_{j+1}-y_i}\cdot\left(y-y_{j+1}\right) \\ \text{for}(\mathrm{x},\mathrm{y})\in \mathrm{T}_2\,. \end{cases} \qquad (22)$$

The polynomial basis $\{P_{i,j}(x,y)\}$ with $i\in\{0,1,2,...,n-1\}$ and $j\in\{0,1,2,...,m-1\}$ of grade one is given by (22) linearly interpolatos on parts function f on the field $D=\bigcup_{i=1}^{n-1}\bigcup_{j=0}^{m-1}D_{i,j}$.

In the particular case when the divisions Δ_1 and Δ_2 are equidistant, namely:

$x_{i+1} - x_i = \frac{b-a}{n} = h,\ y_{j+1} - y_j = \frac{d-c}{m} = k$ and using the divided differences, then, the polynomial $P_{i,j}$ given by (22) may be written in the following manner:

$$P_{i,j}(x,y) = \begin{cases} z_{i,j} + \frac{f[x_i, x_{i+1}; y_j]}{h}(x - x_i) + \frac{f[x_i; y_j, y_{j+1}]}{k}(y - y_j) & \text{for} \quad (x,y) \in T_1 \\ z_{i+1,j+1} + \frac{f[x_i, x_{i+1}; y_{j+1}]}{h}(x - x_{i+1}) + \frac{f[x_{i+1}; y_j, y_{j+1}]}{k}(y - y_{j+1}) & \text{for} \quad (x,y) \in T_2 \end{cases} \tag{23}$$

The error made when the function f replaced by the polynomial $P_{i,j}$ given by (23) is:

$$e_\tau = \left| f(x,y) - P_{i,j}(x,y) \right| \le 2 \cdot M_1 \cdot \left(h^2 + k^2 + h \cdot k \right) \tag{24}$$

where:

$$M_1 = \max_{(x,y)\in D} \left\{ \left| \frac{\partial^2 f}{\partial x^2}(x,y) \right| ; \left| \frac{\partial^2 f}{\partial x \partial y}(x,y) \right| ; \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right| \right\} \tag{25}$$

Conclusions

The paper presents two cases of interpolation of a function with two variable values, with grade one polynomial and determines the linear interpolation on triangular subfields this being the most frquenty used method of interpolation and the interpolation with two grade polynomial on rectangular fields, which are less applied, having in exchange a lesser error.

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Interpolarea Liniară şi Pătratică a Funcţiilor cu Două Variabile cu Noduri Simple

Rezumat

Problema interpolării unei funcţii de două variabile este o problemă care prezintă dificultăţi majore în rezolvare în funcţie de formă, gradul polinomului de interpolare şi de numărul punctelor din domeniul de definiţie. În lucrare sunt prezentate două cazuri de interpolare liniară şi pătratică cu aplicabilitate mare în probleme cu domenii elementare de tip triunghi sau dreptunghi.