BULETINUL Universității Petrol – Gaze din Ploiești	Vol. LIX No. 2/2007	1 - 16	Seria Matematică - Informatică - Fizică

Classes of Infinite Order Pseudodifferential Operators

Mihai Pascu

Universitatea Petrol-Gaze din Ploiești, Bd. București 39, Ploiești, Catedra de Matematică
Institutul de Matematică al Academiei Române
e-mail: Mihai.Pascu@imar.ro

Abstract

Pseudodifferential operators whose symbols have an exponential growth with respect to the phase variable can be defined as operators which act in ultradistribution spaces. We provide here a very short review of some classes of such operators which are defined in Gevrey type ultradistribution spaces and we introduce two classes of infinite order pseudodifferential operators which act in Gelfand-Shilov-Roumieu spaces of tempered ultradistributions.

Key words: *infinite order pseudodifferential operator, tempered ultradistributions*

Introduction

The pseudodifferential operators can be defined by the formula

$$\sigma_\tau(x,D)\varphi(x)=(2\pi)^{-n}\int\int \mathrm{e}^{i<x-y,\xi>}\sigma(\tau x+(1-\tau)y,\xi)\varphi(y)\mathrm{d}y\mathrm{d}\xi\,,$$

for some $\tau\in[0,1]$. The symbols σ and the functions φ belongs to appropriate classes of functions, depending in general on the type of the problem we study.

The most important quantizations are obtained for $\tau = 1$ - the Kohn-Nirenberg quantization:

$$\sigma_{KN}(x,D)\varphi(x)=(2\pi)^{-n}\int \mathrm{e}^{i<x,\xi>}\sigma_{KN}(x,\xi)\hat{\varphi}(\xi)\mathrm{d}\xi\,,$$

where $\hat{\varphi}$ is the Fourier transform of the function (distribution) φ and σ_{KN} is the Kohn-Nirenberg symbol of the operator and for $\tau = 1/2$ - the Weyl quantization:

$$\sigma_W(x,D)\varphi(x)=(2\pi)^{-n}\int\int \mathrm{e}^{i<x-y,\xi>}\sigma_W(\frac{x+y}{2},\xi)\varphi(y)\mathrm{d}y\mathrm{d}\xi\,,$$

where σ_W is the Weyl symbol of the operator.

The pseudodifferential operators are generalizations of differential operators with variable coefficients. Therefore usually one demands that the symbols have polynomial growth with respect to the phase variable ξ. These growth conditions imposed on the symbols are of local nature – polynomial growth with respect to the phase variable ξ of the symbol and of its derivatives uniform with respect to the space variable x in a compact set K, where K is an arbitrary compact set included in an open set Ω or of global nature - polynomial growth with respect to the phase variable ξ of the symbol and of its derivatives uniform with respect to the space variable x in $\boldsymbol{R}^n$.

The local conditions and the Kohn-Nirenberg quantization are especially used in the study of partial differential equations, the global conditions and the Weyl quantization – in quantum mechanics.

Since the theory of ultradistributions and of hyperfunctions was developed, it was also possible to define and study pseudodifferential operators of infinite order.

Our paper has two aims. We shall first make a short review of classes of infinite order pseudodifferential operators studied until now, paying more attention to psudodifferential operators which are defined in classes of ultradistributions (in the next section), The second aim is to introduce a new class of pseudodifferential operators of infinite order acting in spaces of ultradistributions (in the last section). The paper contains also some results on numerical sequences and their associated functions and on spaces of rapidly decreasing functions, needed for the formulation and for the proof of the final results (Theorems 2 and 3).

In general, the letter C will denote an arbitrary constant greater than or equal with 1. But occasionally, especially in the proofs, we shall distinguish between the different constants and in order to do this we shall add subscripts or superscripts to C.

Some Classes of Pseudodifferential Operators of Infinite Order

Even if our main result provides a class of infinite order pseudodifferential operators whose symbols satisfy global conditions, we present at the beginning of this section some classes of pseudodifferential operators of infinite order which symbols satisfy local conditions. These classes were first introduced.

In [1], L. Boutet de Monvel defined a class of analytical pseudodifferential operators of infinite order using weight functions $\Lambda:[0,\infty)\to(0,\infty)$ which are continuous, increasing functions such that

$$\lim_{r\to\infty} e^{-\varepsilon r}\Lambda(r)=0,\ \lim_{r\to\infty} e^{\varepsilon r}\Lambda(r)=+\infty,\ (\forall)\varepsilon>0\,.$$

The symbols are analytical functions $\sigma:\Omega\times\boldsymbol{R}^n\to\boldsymbol{C}$ such that for every compact set $K\subset\Omega$ there exist $\varepsilon, c>0$ such that σ is holomorphic in

$$K(\varepsilon)=\left\{(x,\xi)\in\boldsymbol{C}^n\times\boldsymbol{C}^n;\ d(x,K)<\varepsilon,\ (\mathrm{Im}\xi)^2<\varepsilon\left[(\mathrm{Re}\xi)^2+1\right]\right\}$$

and

$$\left|\sigma(x,\xi)\right|\le c\Lambda(|\xi|),\ (\forall)\ (x,\xi)\in K(\varepsilon)\,.$$

Then $\sigma_{KN}(x,D)u$ is a hyperfunction for every $u\in C_0^\infty(\Omega)$.

More closed to the operators we shall introduce are the ultradifferential operators considered by H. Komatsu in [7] and the pseudodifferential operators of infinite order on Gevrey classes studied by L. Zanghirati in [13].

Before describing these operators, let us shortly recall the definitions of the spaces of ultradifferentiable functions.

Let $(M_p)_{p\in N}$ be a sequence of positive numbers. An infinitely differentiable function φ defined on an open set Ω is called an ultradifferentiable function of class (M_p) (of Beurling type), respectively $\{M_p\}$ (of Roumieu type), if for every compact set $K\subset\Omega$ and every $h>0$ there exists a positive constant C (respectively if for every compact set $K\subset\Omega$ there exist positive constants C and h) such that

$$\left|D^{\alpha}\varphi(x)\right|\le Ch^{|\alpha|}M_{|\alpha|},\ (\forall)\,x\in K,\ (\forall)\alpha\in \boldsymbol{N}^{n}.$$

If $M_p=p^{pr}$ for some $r>1$ (we put $M_0=1$), then one obtains the Gevrey spaces of functions.

If the sequence $(M_p)_p$ is *logarithmic convex* (i.e. if $M_p^2\le M_{p-1}M_{p+1},\ (\forall)p>0$) and satisfies the *non-quasi-analiticity condition* $\sum\limits_{p\ge 1}\dfrac{M_{p-1}}{M_p}<\infty$, then the spaces of ultradifferentiable functions are non-trivial.

A formal sum

$$P(D)=\sum_{|\alpha|\ge 0}a_{\alpha}D^{\alpha},\ a_{\alpha}\in\boldsymbol{C},\ (\forall)|\alpha|\ge 0$$

is called an ultradifferential operator of class (M_p), respectively $\{M_p\}$, if there exist two positive constants L and C (respectively if for every $L>0$ there exists $C>0$) such that

$$|a_{\alpha}|\le\frac{CL^{|\alpha|}}{M_{|\alpha|}},\ (\forall)|\alpha|\ge 0\,.$$

If the sequence $(M_p)_p$ satisfies a third condition, called by Komatsu *stability under ultradifferential operators*, i.e. if there exists a constant C such that

$$M_p\le AH^{p}M_qM_{p-q},\ (\forall)p\ge 0,\ 0\le q\le p\,, \tag{1}$$

then $P(D)$ can be defined as a continuous operator on $\mathsf{D}^{(Mp)}(\Omega)$ (respectively on $\mathsf{D}^{\{Mp\}}(\Omega)$) (and also on the duals of these spaces called ultradistribution spaces).

Remark. The sequences $\left(M_p\right)_p=\left(p^{p\gamma}\right)_p$, which define the Gevrey spaces of functions, satisfy the conditions from above for $\gamma>1$.

L. Zanghirati in [13] considered operators with symbols which are smooth functions $\sigma:\Omega\times\boldsymbol{R}^n\to\boldsymbol{C}$ satisfying the following condition: for every compact set $K\subset\Omega$ there exist positive constants h and B and for every $\varepsilon>0$ there exists a constant $C>0$ such that

$$\left|D_{\xi}^{\alpha}D_{x}^{\beta}\sigma(x,\xi)\right|\le Ch^{|\alpha+\beta|}\alpha!\beta!^{r(\rho-\delta)}(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|}\mathrm{e}^{\varepsilon|\xi|^{1/r}},$$

for every multi-indices α and β, x in Ω and ξ in $\boldsymbol{R}^n$ such that $|\xi|\ge B|\alpha|^{r}$. Here, $r>1,\ 0\le\delta<\rho\le 1,\ r\rho\ge 1$.

Then $\sigma_{KN}(x,D)$ can be defined as a continuous operator defined on the Gevrey space of functions compactly supported $G_0^r(\Omega)$ with values into $G^r(\Omega)$.

We shall describe now two types of pseudodifferential operators of infinite order which symbols satisfy global estimates. But, before doing this, we shall shortly recall the definition of *S – type spaces or Gelfand-Shilov-Roumieu* (*GSR*) spaces. These are subspaces of the Schwartz space of rapidly decreasing functions.

For $(M_p)_p$ and $(N_p)_p$ two logarithmic convex sequences $\mathsf{S}(\{M_p\},\{N_p\})$ is the space of the functions φ which have the property that there exist positive constants C, h and k such that

$$\left|x^\beta D^\alpha \varphi(x)\right| \le Ch^{|\alpha|}k^{|\beta|}M_{|\alpha|}N_{|\beta|},\ (\forall)\, x\in \boldsymbol{R}^n,\ (\forall)\alpha,\beta\in \boldsymbol{N}^n \tag{2}$$

and $\mathsf{S}((M_p),(N_p))$ is the space of the functions φ which have the property that for every positive constants h and k there exists a positive constants C such that

$$\left|x^\beta D^\alpha \varphi(x)\right| \le Ch^{|\alpha|}k^{|\beta|}M_{|\alpha|}N_{|\beta|},\ (\forall)\, x\in \boldsymbol{R}^n,\ (\forall)\alpha,\beta\in \boldsymbol{N}^n .$$

If $(M_p)_p = (N_p)_p$, what we shall assume in what follows, we simply write $\mathsf{S}(\{M_p\},\{M_p\}) = \mathsf{S}(\{M_p\})$ and $\mathsf{S}((M_p),(M_p)) = \mathsf{S}((M_p))$. (If $(M_p)_p = (N_p)_p$ and $(M_p)_p$ satisfies the condition of stability under ultradifferential operators and condition (3) from the beginning of section 3, then the GSR spaces are invariant to the Fourier transform. The dual spaces are denoted with $\mathsf{S}'(\{M_p\})$, respectively $\mathsf{S}'((M_p))$.

In [2], M. Cappiello studied pseudodifferential operators which symbols satisfy the following estimates. For every $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\left|D_\xi^\alpha D_x^\beta \sigma(x,\xi)\right| \le C^{|\alpha+\beta|}(\alpha!)^\mu(\beta!)^\nu(1+|\xi|)^{-|\alpha|}(1+|x|)^{-|\beta|}\mathrm{e}^{\varepsilon(|\xi|^{1/r}+|x|^{1/r})},$$

for every multi-indices α and β, and x and ξ in $\boldsymbol{R}^n$, for some $\mu,\nu,r\in \boldsymbol{R}$ such that

$$\mu > 1,\ \nu > 1,\ r \ge \mu+\nu-1 .$$

Then $\sigma_{KN}(x,D)$ can be defined as a continuous operator in $\mathsf{S}((p^{pr}))$.

The approach used in [13] and [2] is similar to the classical one, developed by L. Hörmander in his pioneering works. A different approach, based on concepts issued from the time-frequency analysis is used by S. Pilipović and N. Teofanov in [11].

Let us shortly describe their approach and the result obtained by them. For $\gamma(=1/r)\in[0,1)$, a continuous function $w:\boldsymbol{R}^n\times\boldsymbol{R}^n\to(0,\infty)$ is called a γ–exp-type weight if there exist $s\ge 0$ and $C > 0$ such that

$$w(x+y,\xi+\eta)\le C\mathrm{e}^{s(|x|^\gamma+|\xi|^\gamma)}w(y,\eta),\ (\forall)x,y,\xi,\eta\in\boldsymbol{R}^n ,$$

i.e. if w is moderate with respect to the weight $\mathrm{e}^{s(|x|^\gamma+|\xi|^\gamma)}$. For $\gamma < 1$ the weight $\mathrm{e}^{s(|x|^\gamma+|\xi|^\gamma)}$ is submultiplicative. For w a γ–exp-type weight, $1\le p,q<\infty$ and t real, the ultra-modulation space $M_{p,q}^{w,t}$ is the space of the ultradistributions $u\in\mathsf{S}'((p^{pr}))$ such that

$$\left[\int\left(\int\left|\left\langle\overline{T_xM_\xi g},u\right\rangle\right|^p w(x,\xi)^p\mathrm{e}^{t(|x|^\gamma+|\xi|^\gamma)}\,\mathrm{d}x\right)^{q/p}\mathrm{d}\xi\right]^{1/q}<\infty .$$

Here, g is an arbitrary window from $\mathcal{S}((p^{pr}))$, T_x is the operator of translation with x and M_ξ is the operator of multiplication with $e^{2\pi i\xi\bullet}$. The function $\left\langle \overline{T_x M_\xi g}, u \right\rangle$ is called in time-frequency analysis the short time Fourier transform of u of window g.

The symbols introduced in [11] are the smooth functions $\sigma : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$ which satisfy the following condition: there exist positive constants h, k, λ, τ and C such that

$$\left|D_\xi^\alpha D_x^\beta \sigma(x,\xi)\right| \le Ch^{|\alpha|}k^{|\beta|}(\alpha!\beta!)^r e^{\lambda|x|^\gamma + \tau|\xi|^\gamma},$$

for every multi-indices α and β, and x and ξ in $\boldsymbol{R}^n$.

One proves that if h and k satisfy some additional technical conditions, then $\sigma_W(x,D) : M_{p,q}^{w,0} \to M_{p,q}^{\widetilde{w},0}$ is a continuous operator for $\widetilde{w}(x,\xi) = w(x,\xi)e^{-2^\gamma \lambda|x|^\gamma - \tau|\xi|^\gamma}$.

The proof is based on the fact that a Wilson basis of exponential decay is an unconditional basis in $M_{p,q}^{w,t}$. Wilson basis are orthonormal basis in L^2 which elements are "simple" linear combinations of time – frequency shifts of a fixed function. More precisely, the elements of a Wilson basis are the functions $T_k g$ with k in $\boldsymbol{Z}$ and the functions $\frac{1}{\sqrt{2}} T_{\frac{k}{2}}(M_n + (-1)^{k+n} M_{-n})g$ with k in $\boldsymbol{Z}$ and n in $\boldsymbol{N}$, for some g in $L^2(\boldsymbol{R}^n)$. A Wilson basis is of exponential decay if both g and its Fourier transform are $\mathcal{O}(e^{-\delta|\cdot|})$ for some positive δ. The proof of the existence of Wilson basis of exponential decay was given in [3]. In their proof, Pilipović and Teofanov study the action of the pseudodifferential operator on the elements of a Wilson basis of exponential decay.

Some Lemmas on Numerical Sequences and their Associated Functions

In this section we shall recall some simple properties of the sequences of numbers and of their associated functions and we shall prove some technical lemmas which we shall use further. Even if finally we shall consider sequences $(M_p)_p$ which are logarithmic convex, satisfy the condition of stability under ultradifferential operators and such that

$$p^\mu M_{p-1} \le CM_p,\ (\forall) p \ge 1 \tag{3}$$

for some positive constant C and some $\mu \ge \frac{1}{2}$, in the hypothesis of the results we shall prove, we shall always mention the minimal conditions which trigger the conclusion. (We could call condition (2) the condition of nontriviality of S-type spaces.)

For simplicity, we shall also assume that $1 = M_0 \le M_1$.

Let us remark that $(M_p)_p = (p^{p\gamma})_p$, satisfies these conditions for $\gamma > 1/2$.

Lemma 1. If $(M_p)_p$ is a logarithmic convex sequence and if $1 = M_0 \le M_1$ then $(M_p)_p$ is an increasing sequence and

$$M_p^2 \le M_{p+q} M_{p-q},\ (\forall) p,q \ge 0,\ q \le p. \tag{4}$$

The proof of the lemma is simple. It can be made by induction.

If $(M_p)_p$ satisfies (3), then we also have that $\lim_{p\to\infty}\sqrt[p]{M_p}=\infty$. Therefore we can define its associated function $M:(0,\infty)\to\boldsymbol{R}$ by the formula

$$M(r)=\sup_{p\geq 0}(p\ln r-\ln M_p),\ (\forall)r>0. \tag{5}$$

Then M is an increasing function and is identically equal with 0 for $r < M_1$. If M is the function associated to some logarithmic convex sequence $(M_p)_p$, then

$$\ln M_p=\sup_{r>0}(p\ln r-M(r)),\ (\forall)p\geq 0. \tag{6}$$

Lemma 2. If $(M_p)_p$ is a logarithmic convex sequence which satisfies (3)*,* if M is its associated function and if $a > 0$*,* then

$$\left|x^{\alpha}\right|\exp(-M(a|x|))\leq a^{|\alpha|}M_{|\alpha|},\ (\forall)\alpha\in\boldsymbol{N}^n,\ (\forall)x\in\boldsymbol{R}^n\setminus\{0\}.$$

Proof. Indeed,

$$\left|x^{\alpha}\right|\exp(-M(a|x|))\leq a^{|\alpha|}M_{|\alpha|},\ (\forall)\alpha\in\boldsymbol{N}^n,\ (\forall)x\in\boldsymbol{R}^n\setminus\{0\},$$

if and only if

$$\ln\left(\left|x^{\alpha}\right|\right)-M(a|x|)\leq|\alpha|\ln a+\ln M_{|\alpha|},\ (\forall)\alpha\in\boldsymbol{N}^n,\ (\forall)x\in\boldsymbol{R}^n\setminus\{0\}.$$

Since $\left|x^{\alpha}\right|\leq|x|^{|\alpha|}$, this last estimate will follow from

$$|\alpha|\ln|x|-M(a|x|)\leq|\alpha|\ln a+\ln M_{|\alpha|},\ (\forall)\alpha\in\boldsymbol{N}^n,\ (\forall)x\in\boldsymbol{R}^n\setminus\{0\},$$

which is equivalent with

$$|\alpha|\ln(a|x|)+\ln M_{|\alpha|}\leq M(a|x|),\ (\forall)\alpha\in\boldsymbol{N}^n,\ (\forall)x\in\boldsymbol{R}^n\setminus\{0\}.$$

This last estimate is an immediate consequence of the definition of the associate function.

Lemma 3. Let $(M_p)_p$ and $(N_p)_p$ be two logarithmic convex sequences such that $\lim_{p\to\infty}\sqrt[p]{M_p}=\lim_{p\to\infty}\sqrt[p]{N_p}=\infty$. Then the sequence $(Q_p)_p$ defined by the formula

$$Q_p=\inf_{0\leq q\leq p}M_{p-q}N_q,\ (\forall)p\geq 0$$

is also logarithmic convex. If *M, N, Q* are the functions associated to $(M_p)_p$, $(N_p)_p$ and $(Q_p)_p$ respectively, then

$$Q(r)=M(r)+N(r),\ (\forall)r>0.$$

Proof. The first assertion is proved in [12] (Lemma 4 from chapter 1). The second one is also well known. You can find its straightforward proof in [10].

Lemma 4. If $(M_p)_p$ *and* $(N_p)_p$ are two logarithmic convex sequences such that $\lim_{p\to\infty}\sqrt[p]{M_p} = \lim_{p\to\infty}\sqrt[p]{N_p} = \infty$ and if *M* and *N* are their associated functions, then

$$M_p \le N_p,\ (\forall)p \ge 0$$

if and only if

$$M(r) \ge N(r),\ (\forall)r > 0.$$

This lemma is a direct consequence of formulas (4) and (5).

Lemma 5. If $(M_p)_p$ is a logarithmic convex sequence such that $\lim_{p\to\infty}\sqrt[p]{M_p} = \infty$ and which satisfies the condition of stability under ultradifferential operators and if *M* is its associated function, then

$$M(br) + M(cr) \le M(ar),\ (\forall)b, c, r > 0, \tag{7}$$

for $a = C\max(b,c)$, where *C* is the constant from (1).

Proof. We first notice that the sequence which associated function is $M(a\cdot)$ has the general term equal with $a^{-p}M_p$. Using Lemma 3, we deduce that the sequence corresponding to the function $M(b\cdot) + M(c\cdot)$ has the general term equal with

$$\inf_{0\le q\le p} b^{-q}M_q c^{p-q}M_{p-q}.$$

Therefore, in order to have (7), it is sufficient, accordingly to Lemma 4, to choose *a* such that

$$a^{-p}M_p \le b^{-q}M_q c^{p-q}M_{p-q},\ (\forall)p, q \ge 0,\ q \le p.$$

The number $a = C\max(b,c)$, where *C* is the constant from (1), is a good choice.

Lemma 6. If $(M_p)_p$ is a logarithmic convex sequence which satisfies condition (3), then

$$(r!)^{1/2} M_{p-r} \le C^r M_p,\ (\forall)r, p \ge 0,\ r \le p,$$

where *C* is the constant from (3).

The proof is straightforward and can be made by induction on *r*.

Corrolary. If $(M_p)_p$ is a logarithmic convex sequence which satisfies condition (3), then

$$r!M_{p-r}M_{q-r} \le C^{2r}M_pM_q,\ (\forall)r, p, q \ge 0,\ r \le \min(p,q),$$

where *C* is the constant from (3).

Lemma 7. If $(M_p)_p$ is a logarithmic convex sequence which satisfies (3) with $\mu > \frac{1}{2}$, then $(\forall)\varepsilon > 0,\ (\exists)C_\varepsilon > 0$ such that

$$(r!)^{1/2} M_{p-r} \le C_\varepsilon \varepsilon^r M_p,\ (\forall)r, p \ge 0,\ r \le p.$$

Proof. For every $\varepsilon > 0$, there exists some positive constant C_ε such that

$$(r!)^{1/2} M_{p-r} \le C_\varepsilon \varepsilon^r (r!)^\mu M_{p-r} \le C_\varepsilon (C\varepsilon)^r M_p,\ (\forall) r, p \ge 0,\ r \le p,$$

where C is the constant from (3).

Corollary. If $(M_p)_p$ is a logarithmic convex sequence which satisfies (3) with $\mu > \frac{1}{2}$, then $(\forall)\varepsilon > 0,\ (\exists) C_\varepsilon > 0$ such that

$$r! M_{p-r} M_{q-r} \le C_\varepsilon \varepsilon^{2r} M_p M_q,\ (\forall) r, p, q \ge 0,\ r \le \min(p, q).$$

The Spaces *S* ({*M_p*}) and *S* ((*M_p*))

In this section we shall always assume that $(M_p)_p$ is a logarithmic convex sequence which satisfies the condition of stability under ultradifferential operators and the condition (3).

We have already introduced the space $\mathsf{S}(\{M_p\})$ in the section 2. If $(M_p)_p$ satisfies the conditions mentioned above, then $\mathsf{S}(\{M_p\})$ is nontrivial and the Fourier transform is a topological isomorphism on $\mathsf{S}(\{M_p\})$ (see e.g. [10]). We shall prove that the Fourier transform is a topological isomorphism on $\mathsf{S}((M_p))$ also if condition (3) is fulfilled with some $\mu > \frac{1}{2}$.

For the beginning, we give another description of the functions which belong to $\mathsf{S}(\{M_p\})$ and $\mathsf{S}((M_p))$.

Lemma 8. Let M be the function associated to a sequence $(M_p)_p$ and let φ be a smooth function defined on $\boldsymbol{R}^n$.

i) If

$$\left| x^\beta \partial^\alpha \varphi(x) \right| \le C A^{|\alpha|} B^{|\beta|} M_{|\alpha|} M_{|\beta|},\ (\forall) x \in \boldsymbol{R}^n,\ (\forall) \alpha, \beta \in \boldsymbol{N}^n,$$

then

$$\left| \partial^\alpha \varphi(x) \right| \exp\left(M\left(\frac{|x|}{\delta B \sqrt{n}} \right) \right) \le \frac{C}{(\delta - 1)^n} A^{|\alpha|} M_{|\alpha|},\ (\forall) x \in \boldsymbol{R}^n,\ (\forall) \alpha \in \boldsymbol{N}^n,\ (\forall) \delta > 1.$$

ii) If

$$\left| \partial^\alpha \varphi(x) \right| \exp\left(M\left(\frac{|x|}{B} \right) \right) \le C A^{|\alpha|} M_{|\alpha|},\ (\forall) x \in \boldsymbol{R}^n,\ (\forall) \alpha \in \boldsymbol{N}^n, \tag{8}$$

then

$$\left| x^\beta \partial^\alpha \varphi(x) \right| \le C A^{|\alpha|} B^{|\beta|} M_{|\alpha|} M_{|\beta|},\ (\forall) x \in \boldsymbol{R}^n,\ (\forall) \alpha, \beta \in \boldsymbol{N}^n.$$

Proof. We shall follow [12].

i) It is clear that

$$M(r) \le \ln\left(\sum_{q\ge 0} \frac{r^q}{M_q}\right), (\forall) r > 0 .$$

Hence

$$\exp(M(r)) \le \sum_{q\ge 0} \frac{r^q}{M_q}, (\forall) r > 0 .$$

On the other hand, if we denote $\|x\|_\infty = \max_i |x_i|, (\forall) x = (x_1,...,x_n) \in \boldsymbol{R}^n$, for $\delta > 1$ we have that

$$\left|\partial^\alpha \varphi(x)\right| \sum_{q\ge 0}\left(\frac{|x|}{\delta B\sqrt{n}}\right)^q \cdot \frac{1}{M_q} \le \left|\partial^\alpha \varphi(x)\right| \sum_{q\ge 0}\left(\frac{\|x\|_\infty}{\delta B}\right)^q \cdot \frac{1}{M_q} \le$$

$$\le \left|\partial^\alpha \varphi(x)\right| \sum_{\beta\ge 0} \frac{\left|x^\beta\right|}{(\delta B)^{|\beta|}} \cdot \frac{1}{M_{|\beta|}} \le CA^{|\alpha|} M_{|\alpha|} \sum_{\beta \ge 0} \delta^{-|\beta|} = \frac{C}{(\delta-1)^n} A^{|\alpha|} M_{|\alpha|} .$$

ii) This point is obtained by observing that

$$\frac{r^q}{M_q} \le \mathrm{e}^{M(r)}, (\forall) r > 0, (\forall) q \in \boldsymbol{N} ,$$

and replacing r with $\frac{|x|}{B}$.

We can introduce an inductive limit topology on $\mathsf{S}(\{M_p\})$ and a projective limit topology on $\mathsf{S}((M_p))$, using either (2), either (8) to define norms on subspaces of $\mathsf{S}(\{M_p\})$, respectively on spaces which intersection is $\mathsf{S}((M_p))$. The topologies such introduced will coincide.

As an example, if we use (2), then we put for $A, B > 0$

$$\|\varphi\|_{A,B} = \sup_{\alpha,\beta\ge 0} \sup_{x\in \boldsymbol{R}^n} \left|x^\beta \partial^\alpha \varphi(x)\right| A^{-|\alpha|} B^{-|\beta|} (M_{|\alpha|} M_{|\beta|})^{-1}$$

and

$$\mathsf{S}_{A,B}(\{M_p\}) = \{\varphi \in \mathsf{S}(\{M_p\}); \|\varphi\|_{A,B} < \infty\} .$$

Then $\left(\mathsf{S}_{A,B}(\{M_p\}), \|\ \|_{A,B}\right)$ is a Banach subspace of $\mathsf{S}(\{M_p\})$ and

$$\mathsf{S}(\{M_p\}) = \bigcup_{A,B>0} \mathsf{S}_{A,B}(\{M_p\}).$$

Remark that the space $\mathsf{S}_{A,B}(\{M_p\})$ is continuously embedded in $\mathsf{S}_{A',B'}(\{M_p\})$ if $A' \ge A$, $B' \ge B$. We can introduce a topology on $\mathsf{S}(\{M_p\})$ by mentioning what a sequence converges to 0 ([GS]) means. We say that a sequence $(\varphi_j)_j \subset \mathsf{S}(\{M_p\})$ converges to 0 if there exist A, $B > 0$ such that $\varphi_j \in \mathsf{S}_{A,B}(\{M_p\})$, $(\forall) j > 0$ and if $\varphi_j \underset{j\to\infty}{\to} 0$ in $\mathsf{S}_{A,B}(\{M_p\})$.

Then a linear operator $A : \mathsf{S}_{A,B}(\{M_p\}) \to \mathsf{S}_{A,B}(\{M_p\})$ will be continuous if for every A, $B >$ 0, there exist A', B' > 0 such that $A\big(\mathsf{S}_{A,B}(\{M_p\})\big) \subset \mathsf{S}_{A',B'}(\{M_p\})$ and if $A : \mathsf{S}_{A,B}(\{M_p\}) \to \mathsf{S}_{A',B'}(\{M_p\})$ is continuous.

We shall give now another technical lemma which will be used in the next section.

Lemma 9. If A, $B > 0$ then there exist two positive constants A', B' which depend continuously on A and B and for every multi-index γ there exists a positive constant $C'(\gamma)$ which depends also continuously on B such that

$$\left|x^\gamma \partial^\alpha (x^\beta \varphi(x))\right| \le C'(\gamma)(A')^{|\alpha|}(B')^{|\beta|} M_{|\alpha|} M_{|\beta|} \exp(-M(|x|/B'))\|\varphi\|_{A,B}$$

for every multi-indices α,β, for every x in $\boldsymbol{R}^n$ and for every $\varphi \in \mathsf{S}_{A,B}(\{M_p\})$.

Proof. Let us consider first the case when $\beta = 0$. We apply Lemma 7, i) with $\delta = 2$ and obtain that

$$\left|\partial^\alpha \varphi(x)\right| \exp(M(a|x|) \le \|\varphi\|_{A,B} A^{|\alpha|} M_{|\alpha|},\ (\forall) x \in \boldsymbol{R}^n,\ (\forall)\alpha \ge 0$$

for $a = \dfrac{1}{2B\sqrt{n}}$.

If C is the constant from (1), then, using Lemma 2 and Lemma 5, we have for $x \ne 0$ that

$$\left|x^\gamma \partial^\alpha \varphi(x)\right| = \left|x^\gamma \exp\left(-M\left(\frac{a}{C}|x|\right)\right)\right| \left|\exp\left(M\left(\frac{a}{C}|x|\right)\right)\partial^\alpha \varphi(x)\right| \le$$

$$\le \left|x^\gamma \exp\left(-M\left(\frac{a}{C}|x|\right)\right)\right| \exp\left(-M\left(\frac{a}{C}|x|\right)\right) \left|\exp\big(M(a|x|)\big)\partial^\alpha \varphi(x)\right| \le$$

$$\le \left(\frac{C}{a}\right)^{|\gamma|} M_{|\gamma|} \left|\exp\big(M(a|x|)\big)\partial^\alpha \varphi(x)\right| \exp\left(-M\left(\frac{a}{C}|x|\right)\right) \le$$

$$\le \|\varphi\|_{A,B} \left(\frac{C}{a}\right)^{|\gamma|} M_{|\gamma|} A^{|\alpha|} M_{|\alpha|} \exp\left(-M\left(\frac{a}{C}|x|\right)\right).$$

Therefore, in this case, we may take $A' = A$, $B' = 2CB\sqrt{n}$ and $C'(\gamma) = \left(\dfrac{C}{a}\right)^{|\gamma|} M_{|\gamma|}$.

In the general case we have

$$\left|x^\gamma \partial^\alpha (x^\beta \varphi(x))\right| \le \sum_{\alpha' \le \alpha,\ \alpha' \le \beta} \binom{\alpha}{\alpha'}\binom{\beta}{\alpha'}(\alpha')! \left|x^{\gamma+\beta-\alpha'} \partial^{\alpha-\alpha'} \varphi(x)\right| \le$$

$$\le \sum_{\alpha' \le \alpha,\ \alpha' \le \beta} \binom{\alpha}{\alpha'}\binom{\beta}{\alpha'}(\alpha')! \left(\frac{C}{a}\right)^{|\gamma+\beta-\alpha'|} M_{|\gamma+\beta-\alpha'|} A^{|\alpha-\alpha'|} M_{|\alpha-\alpha'|} \exp\left(-M\left(\frac{a}{C}|x|\right)\right)\|\varphi\|_{A,B}\ .$$

Using Lemma 6, we see that

$$(\alpha')! M_{|\gamma+\beta-\alpha'|} M_{|\alpha-\alpha'|} \le C_1^{2|\alpha'|} M_{|\gamma+\beta|} M_{|\alpha|},$$

where C_1 is the constant appearing in the estimate (3). Therefore

$$\left|x^{\gamma}\partial^{\alpha}(x^{\beta}\varphi(x))\right| \le$$

$$\le \sum_{\alpha'\le\alpha,\ \alpha'\le\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\alpha'}\left(\frac{C}{a}\right)^{|\gamma+\beta-\alpha'|} C_1^{2|\alpha'|}M_{|\gamma+\beta|}A^{|\alpha-\alpha'|}M_{|\alpha|}\exp\left(-M\left(\frac{a}{C}|x|\right)\right)\|\varphi\|_{A,B} \le$$

$$\le 2^{|\alpha+\beta|}\left(\frac{C}{a}\right)^{|\gamma|}\left[\max\left(C_1,\frac{C}{a}\right)\right]^{|\beta|}\left[\max(A,C_1)\right]^{|\alpha|}C^{|\gamma+\beta|}M_{|\gamma|}M_{|\beta|}M_{|\alpha|}\cdot$$

$$\cdot\exp\left(-M\left(\frac{a}{C}|x|\right)\right)\|\varphi\|_{A,B} \le$$

$$\le\left[C^{|\gamma|}\left(\frac{C}{a}\right)^{|\gamma|}M_{|\gamma|}\right]\left[2C\max\left(C_1,\frac{C}{a}\right)\right]^{|\beta|}\left[2\max(A,C_1)\right]^{|\alpha|}M_{|\beta|}M_{|\alpha|}\cdot$$

$$\cdot\exp\left(-M\left(\frac{a}{C}|x|\right)\right)\|\varphi\|_{A,B}.$$

Hence we may take

$$C'(\gamma)=C^{|\gamma|}\left(\frac{C}{a}\right)^{|\gamma|}M_{|\gamma|},$$

$$B'=\max\left[2C\max\left(C_1,\frac{C}{a}\right),\frac{C}{a}\right],$$

$$A'=2\max(A,C_1).$$

Lemma 9′. If condition (3) is fulfilled with some $\mu>\frac{1}{2}$ and if A, $B>0$, then there exist two positive constants A', B' which are $O(A)$, respectively $O(B)$ and for every multi-index γ there exists a positive constant $C'(\gamma)$ which depends also on A and B such that

$$\left|x^{\gamma}\partial^{\alpha}(x^{\beta}\varphi(x))\right|\le C'(\gamma)(A')^{|\alpha|}(B')^{|\beta|}M_{|\alpha|}M_{|\beta|}\exp(-M(|x|/B'))\|\varphi\|_{A,B}$$

for every multi-indices α,β, for every x in $\boldsymbol{R}^n$ and for every $\varphi\in \mathcal{S}_{A,B}(\{M_p\})$.

Proof. The proof is similar to the proof of Lemma 9.

In the case $\beta=0$, as we saw we may take $A'=A$, $B'=2CB\sqrt{n}$ and $C'(\gamma)=\left(\frac{C}{a}\right)^{|\gamma|}M_{|\gamma|}$.

These constants have the properties stated in the conclusion of the lemma.

Differences in the proof appear when we consider the general case. We have, as in the proof of Lemma 9, that

$$\left|x^{\gamma}\partial^{\alpha}(x^{\beta}\varphi(x))\right| \le$$

$$\le \sum_{\alpha'\le\alpha,\ \alpha'\le\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\alpha'}(\alpha')!\left(\frac{C}{a}\right)^{|\gamma+\beta-\alpha'|}M_{|\gamma+\beta-\alpha'|}A^{|\alpha-\alpha'|}M_{|\alpha-\alpha'|}\exp\left(-M\left(\frac{a}{C}|x|\right)\right)\|\varphi\|_{A,B}.$$

Using Lemma 7, we see that $(\forall)\varepsilon>0$, $(\exists)C_{\varepsilon}>0$ such that

$$(\alpha')! M_{|\gamma+\beta-\alpha'|} M_{|\alpha-\alpha'|} \le C_\varepsilon \varepsilon^{2|\alpha'|} M_{|\gamma+\beta|} M_{|\alpha|} .$$

Therefore

$$\left| x^\gamma \partial^\alpha (x^\beta \varphi(x)) \right| \le$$

$$\le C_\varepsilon \sum_{\alpha' \le \alpha,\, \alpha' \le \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\alpha'} \left(\frac{C}{a} \right)^{|\gamma+\beta-\alpha'|} \varepsilon^{2|\alpha'|} M_{|\gamma+\beta|} A^{|\alpha-\alpha'|} M_{|\alpha|} \exp\left(-M\left(\frac{a}{C}|x| \right) \right) \|\varphi\|_{A,B} \le$$

$$\le C_\varepsilon 2^{|\alpha+\beta|} \left(\frac{C}{a} \right)^{|\gamma|} \left[\max\left(\varepsilon, \frac{C}{a} \right) \right]^{|\beta|} [\max(A,\varepsilon)]^{|\alpha|} C^{|\gamma+\beta|} M_{|\gamma|} M_{|\beta|} M_{|\alpha|} \cdot$$

$$\cdot \exp\left(-M\left(\frac{a}{C}|x| \right) \right) \|\varphi\|_{A,B} \le$$

$$\le \left[C_\varepsilon C^{|\gamma|} \left(\frac{C}{a} \right)^{|\gamma|} M_{|\gamma|} \right] \left[2C \max\left(\varepsilon, \frac{C}{a} \right) \right]^{|\beta|} [2\max(A,\varepsilon)]^{|\alpha|} M_{|\beta|} M_{|\alpha|} \cdot$$

$$\cdot \exp\left(-M\left(\frac{a}{C}|x| \right) \right) \|\varphi\|_{A,B} .$$

Hence we may take

$$C'(\gamma) = C_\varepsilon C^{|\gamma|} \left(\frac{C}{a} \right)^{|\gamma|} M_{|\gamma|} ,$$

$$B' = \max\left[2C \max\left(\varepsilon, \frac{C}{a} \right), \frac{C}{a} \right],$$

$$A' = 2\max(A,\varepsilon)$$

for $\varepsilon = \min(A,B)$.

Theorem 1. If condition (3) is fulfilled with some $\mu > \frac{1}{2}$, then the Fourier transform is a topological isomorphism on $\mathsf{S}((M_p))$.

Proof. For simplicity, we shall give the proof for the case $n = 1$. The proof is similar to the proof provided in [9] of the similar statement in the case of $\mathsf{S}(\{M_p\})$. We shall use the fact that the topology on $\mathsf{S}((M_p))$ can be also defined by using L^2 norms instead of L^∞ norms ([12]). It is well known that the Fourier transform is unitary on L^2 (theorem of Plancherel). Therefore, for any φ belonging to $\boldsymbol{S}((M_p))$ and for every positive constants A and B we have that

$$\left\| \xi^q (\hat{\varphi})^{(p)}(\xi) \right\|_2 = \left\| (x^p \varphi(x))^{(q)} \right\|_2 \le$$

$$\le \sum_{r \le \min(p,q)} \frac{q(q-1)\dots(q-r+1)p(p-1)\dots(p-r+1)}{r!} \left\| x^{p-r} \varphi^{(q-r)}(x) \right\|_2 \le$$

$$\le C \sum_{r \le \min(p,q)} \frac{q(q-1)\dots(q-r+1)p(p-1)\dots(p-r+1)}{r!} A^{q-r} B^{p-r} M_{q-r} N_{p-r}$$

for some positive constant C.

If condition (3) is verified, then accordingly to Lemma 7, we see that for every $\varepsilon > 0$ there exists a constant C_ε such that the norm to be estimated is dominated by

$$C_\varepsilon M_q N_p \sum_{r\le\min(p,q)} \frac{q(q-1)...(q-r+1)p(p-1)...(p-r+1)}{(r!)^2}\varepsilon^{2r}A^{q-r}B^{p-r} \le$$
$$\le C_\varepsilon A_1^q B_1^p M_q N_p 2^{p+q} = C_\varepsilon A_2^q B_2^p M_q N_p .$$

We have put, successively, $A_1 = \max(A,\varepsilon)$, $B_1 = \max(B,\varepsilon)$, $A_2 = 2A_1$, $B_2 = 2B_1$.

Therefore if $\varphi \in \boldsymbol{S}((M_p))$, then $\hat{\varphi} \in \boldsymbol{S}((M_p))$. The continuity of the Fourier transform will follow from the closed graph theorem.

Infinite Order Pseudodifferential Operators in $\boldsymbol{S}$ – Type Spaces

We shall say that a function $\sigma \in C^\infty(\boldsymbol{R}^{2n};\boldsymbol{C})$ is in $S^m_{(M_p)}$ if for every $\varepsilon > 0$ there exist two positive constants C_ε, A_ε such that

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)\right| \le C_\varepsilon A_\varepsilon^{|\alpha+\beta|} M_{|\alpha|} M_{|\beta|} (1+|\xi|)^m \exp(M(\varepsilon|\xi|)),\ (\forall)x,\xi \in \boldsymbol{R}^n,\ (\forall)\alpha,\beta \in \boldsymbol{N}^n .$$

Remark 1. If there exist two positive constants c and δ such that $M(r) \ge cr^\delta$, $(\forall)r > 0$, then $S^m_{(M_p)}$ does not depend on m (this is a consequence of Lemma 5). This is the case if $M_p = p^{p\theta}$ for some $\theta > 0$. Then $M(r)$ is equivalent with $r^{1/\theta}$ and we may take $\delta = \dfrac{1}{\theta}$.

For $\sigma \in S^m_{(M_p)}$ we define a pseudodifferential operator $\sigma(x,D)$ using the Kohn – Nirenberg quantization:

$$\sigma(x,D)u(x) = (2\pi)^{-n}\int \mathrm{e}^{i<x,\xi>}\sigma(x,\xi)\hat{\varphi}(\xi)\mathrm{d}\xi,\ (\forall)x \in \boldsymbol{R}^n .$$

Theorem 2. If $(M_p)_p$ is a logarithmic convex sequence which satisfy the condition of stability under ultradifferential operators and condition (3) and if $\sigma \in S^m_{(M_p)}$, then $\sigma(x,D) : \mathsf{S}(\{M_p\}) \to \mathsf{S}(\{M_p\})$ is a continuous operator.

Proof. In our hypothesis, the Fourier transform is a topological isomorphism of $\mathsf{S}(\{M_p\})$ (see e.g. [9]). Therefore it is sufficient to prove that the operator A defined by the formula

$$Au(x) = \int \mathrm{e}^{i<x,\xi>}\sigma(x,\xi)\psi(\xi)\mathrm{d}\xi,\ (\forall)x \in \boldsymbol{R}^n$$

is a continuous operator in $\mathsf{S}(\{M_p\})$.

Let us assume that $\psi \in \mathsf{S}_{A,B}(\{M_p\})$. Then, applying Lemma 9, we obtain that

$$\left|x^\beta \partial_x^\alpha A\psi(x)\right| \le \left|\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\int (i\xi)^{\alpha-\alpha'}(\partial_\xi^\beta \mathrm{e}^{i<x,\xi>})\partial_x^{\alpha'}\sigma(x,\xi)\psi(\xi)\mathrm{d}\xi\right| \le$$

$$\leq \sum_{\alpha'\leq\alpha,\beta'\leq\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\left|\int \mathrm{e}^{i<x,\xi>}(\partial_\xi^{\beta'}\partial_x^{\alpha'}\sigma(x,\xi))\partial_\xi^{\beta-\beta'}(\xi^{\alpha-\alpha'}\psi(\xi))\mathrm{d}\xi\right| \leq$$

$$\leq \sum_{\alpha'\leq\alpha,\beta'\leq\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\int \left|\partial_\xi^{\beta'}\partial_x^{\alpha'}\sigma(x,\xi)\right|(1+|\xi|)^{-m}(1+|\xi|)^{-n-1}\left|\partial_\xi^{\beta-\beta'}(\xi^{\alpha-\alpha'}\psi(\xi))\right|\cdot$$

$$\cdot(1+|\xi|)^{m+n+1}\mathrm{d}\xi \leq$$

$$\leq C_\varepsilon C''\|\psi\|_{A,B}\sum_{\alpha'\leq\alpha,\beta'\leq\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}A_\varepsilon^{|\alpha'+\beta'|}M_{|\alpha'|}M_{|\beta'|}(A')^{|\alpha-\alpha'|}(B')^{|\beta-\beta'|}M_{|\alpha-\alpha'|}M_{|\beta-\beta'|}\cdot$$

$$\cdot\int(1+|\xi|)^{-n-1}\exp(M(\varepsilon|\xi|))\exp\left(-M\left(\frac{|\xi|}{B'}\right)\right)\mathrm{d}\xi .$$

Here C'' is a constant which depends only on $m+n+1$ and B and A' and B' are the constants from Lemma 9.

Therefore, taking $\varepsilon = \frac{1}{B'}$ and using again the fact that $(M_p)_p$ is logarithmic convex, we have that

$$\left|x^\beta\partial_x^\alpha A\psi(x)\right| \leq C_\varepsilon\left(\int(1+|\xi|)^{\text{-n-1}}\mathrm{d}\xi\right)2^{|\alpha+\beta|}[\max(A_\varepsilon,A')]^{|\alpha|}[\max(A_\varepsilon,B')]^{|\beta|}M_{|\alpha|}M_{|\beta|}\|\psi\|_{A,B}.$$

Hence

$$A:\mathsf{S}_{A,B}(\{M_p\})\to\mathsf{S}_{A'',B''}(\{M_p\})$$

is a continuous operator, if $A''=2\max(A_\varepsilon,A')$, $B''=2\max(A_\varepsilon,B')$. The proof is complete.

Remark 2. If σ satisfies the weaker estimates

$$\left|\partial_\xi^\alpha\partial_x^\beta\sigma(x,\xi)\right| \leq C_\varepsilon A_\varepsilon^{|\alpha+\beta|}M_{|\alpha|}M_{|\beta|}(1+|\xi|)^{m+\rho(|\alpha|+|\beta|)}\exp(M(\varepsilon|\xi|)),$$
$$(\forall)x,\xi\in\boldsymbol{R}^n,\ (\forall)\alpha,\beta\in\boldsymbol{N}^n,$$

for some positive ρ and if M is as in Remark 1, then $\sigma(x,D):\mathsf{S}(\{M_p\})\to\mathsf{S}(\{N_p\})$ is a continuous operator for $N_p = p^{\left(\frac{\rho}{\delta}+\eta\right)p}M_p$ with $\eta>0$ arbitrary small.

We shall say that a function $\sigma\in C^\infty(\boldsymbol{R}^{2n};\boldsymbol{C})$ is in $S^m_{\{M_p\}}$ if for every positive constant k there exist two positive constants C and h_k such that

$$\left|\partial_\xi^\alpha\partial_x^\beta\sigma(x,\xi)\right| \leq C_k k^{|\alpha+\beta|}M_{|\alpha|}M_{|\beta|}(1+|\xi|)^m\exp(M(h_k|\xi|)),$$
$$(\forall)x,\xi\in\boldsymbol{R}^n,\ (\forall)\alpha,\beta\in\boldsymbol{N}^n.$$

For $\sigma\in S^m_{\{M_p\}}$ we define a pseudodifferential operator $\sigma(x,D)$ using the Kohn – Nirenberg quantization:

$$\sigma(x,D)u(x)=(2\pi)^{-n}\int\mathrm{e}^{i<x,\xi>}\sigma(x,\xi)\hat{\varphi}(\xi)\mathrm{d}\xi,\ (\forall)x\in\boldsymbol{R}^n.$$

Theorem 3. If $(M_p)_p$ is a logarithmic convex sequence which satisfy the condition of stability under ultradifferential operators and condition (3) and if $\sigma\in S^m_{\{M_p\}}$, then

$$\sigma(x,D):\mathrm{S}((M_p))\to\mathrm{S}((M_p))$$

is a continuous operator.

Proof. In our hypothesis, accordingly to Theorem 1, the Fourier transform is a topological isomorphism of $\mathrm{S}((M_p))$. Therefore it is sufficient to prove that the operator A defined by the formula

$$Au(x)=\int \mathrm{e}^{i<x,\xi>}\sigma(x,\xi)\psi(\xi)\mathrm{d}\xi,\ (\forall)x\in\boldsymbol{R}^n$$

is a continuous operator in $\mathrm{S}((M_p))$.

Let us assume that $\psi\in\mathrm{S}((M_p))$. Then, for every A, $B>0$, applying Lemma 9′, we see that

$$\left|x^{\beta}\partial_x^{\alpha}A\psi(x)\right|\le\left|\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\int(i\xi)^{\alpha-\alpha'}(\partial_{\xi}^{\beta}\mathrm{e}^{i<x,\xi>})\partial_x^{\alpha'}\sigma(x,\xi)\psi(\xi)\mathrm{d}\xi\right|\le$$

$$\le\sum_{\alpha'\le\alpha,\beta'\le\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\left|\int\mathrm{e}^{i<x,\xi>}(\partial_{\xi}^{\beta'}\partial_x^{\alpha'}\sigma(x,\xi))\partial_{\xi}^{\beta-\beta'}(\xi^{\alpha-\alpha'}\psi(\xi))\mathrm{d}\xi\right|\le$$

$$\le\sum_{\alpha'\le\alpha,\beta'\le\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\int\left|\partial_{\xi}^{\beta'}\partial_x^{\alpha'}\sigma(x,\xi)\right|(1+|\xi|)^{-m}(1+|\xi|)^{-n-1}\left|\partial_{\xi}^{\beta-\beta'}(\xi^{\alpha-\alpha'}\psi(\xi))\right|\cdot$$

$$\cdot(1+|\xi|)^{m+n+1}\mathrm{d}\xi\le$$

$$\le C_kC''\|\psi\|_{A,B}\sum_{\alpha'\le\alpha,\beta'\le\beta}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}k^{|\alpha'+\beta'|}M_{|\alpha'|}M_{|\beta'|}(A')^{|\alpha-\alpha'|}(B')^{|\beta-\beta'|}M_{|\alpha-\alpha'|}M_{|\beta-\beta'|}\cdot$$

$$\cdot\int(1+|\xi|)^{-n-1}\exp(M(h_k|\xi|))\exp\left(-M\left(\frac{|\xi|}{B'}\right)\right)\mathrm{d}\xi\,.$$

Here C'' is a constant which depends only on $m+n+1$, A and B and A' and B' are the constants from Lemma 9′.

Therefore, choosing A and B such that $h_k\le\dfrac{1}{B'}$ and using again the fact that $(M_p)_p$ is logarithmic convex, we have that

$$\left|x^{\beta}\partial_x^{\alpha}A\psi(x)\right|\le$$
$$\le C(k,A,B)\left(\int(1+|\xi|)^{-n-1}\mathrm{d}\xi\right)2^{|\alpha+\beta|}[\max(k,A')]^{|\alpha|}[\max(k,B')]^{|\beta|}M_{|\alpha|}M_{|\beta|}\|\psi\|_{A,B}\,.$$

Hence $A\psi\in\mathrm{S}((M_p))$. The continuity will follow from the closed graph theorem.

References

1. B o u t e t d e M o n v e l , L . - Opérateurs pseudo-differentiels analytiques et opérateurs d'ordre infini, , *Annales de l'Institut Fourier de Grenoble*, 223, pp 229-268, 1972
2. C a p i e l l o , M . - Pseudodifferential parametrices of infinite order for SG-hyperbolic problems, *Rendiconti di Seminario Matematica della Universita Politecnica di Torino*, 61, 42003, pp. 411-441, 2003
3. D a u b e c h i e s , I . , J a f f a r d , S . , J o u r n é , J . L . - A simple Wilson orthonormal basis with exponential decay, *SIAM Journal of Mathematical Analysis*, 22(2), pp. 554-573, 1991

4. Gelfand, I.M., Shilov, G.E. - *Generalized functions*, vol. 2, Academic Press, New York-London, 1967
5. Grőchenig, K. - *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2001
6. Liess, O., Rodino, L. - Inhomogeneous Gevrey classes and related pseudo-differential operators, *Bolletino della Unione Matematica Italiana, Analisi Funzionali e Applicazioni*, Serie VI, vol. III.C, no. 1 / 1984, pp. 233-323, 1984
7. Komatsu, H. - Ultradistributions I, structure theorems and characterization, *Journal of the Faculty of Science, University of Tokyo*, Sect. 1A, 20, pp. 25-105, 1973
8. Pascu, M. - Hypoelliptic operators in Denjoy-Carleman classes, *Revue Roumaine de Mathématiques Pures et Appliquées*, 30, 2, pp. 131-145, 1985
9. Pascu, M. - Fourier transform on S type spaces, *Buletinul Universităţii.Petrol-Gaze din Ploieşti, Seria Matematică-Informatică - Fizică*, vol. LVII, 1, pp. 34-38, 2005
10. Pascu M.- Spaces of rapidly decreasing functions, *Buletinul Universităţii.Petrol-Gaze din Ploieşti, Seria Matematică-Informatică - Fizică*, vol. LVII, 2, pp. 14-19, 2005
11. Pilipovič, St., Teofanov, N. - Pseudodifferential operators on ultra-modulation spaces, *Journal of Functional Analysis*, 208, pp. 194-228, 2004
12. Roumieu, Ch. - Sur quelques extensions de la notion de distribution, *Annales Sciéntifiques de l'École Normale Supérieure*, 77, pp. 47-121, 1960
13. Zangiratti. L. - Pseudodifferential operators of infinite order and Gevrey classes, *Annali della Universita di Ferrara* Sez. VII – Sc. Mat., XXXI, pp. 197-219, 1985

Clase de Operatori Pseudodiferenţiali de Ordin Infinit

Rezumat

Operatorii pseudodiferenţiali ale căror simboluri au creştere exponenţială în raport cu variabila de fază la infinit pot fi definiţi ca operatori care acţionează în spaţii de ultradistribuţii. Facem aici o scurtă trecere în revistă a unor clase de astfel de operatori care acţionează în spaţii de ultradistribuţii de tip Gevrey şi introducem două clase de astfel de operatori care acţionează în spaţii de ultradistribuţii temperate de tip Gelfand-Shilov-Roumieu.