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Multidimensional Generalized Riemann Integral. Integrability Criteria

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Abstract

In a previous work (M. Pascu, On the definition of the Multidimensional Riemann Integral, Buletinul Universităţii Petrol-Gaze din Ploieşti, vol. LVIII, No. 2/2006, 99. 9-16) we proposed a general definition of the multidimensional generalized Riemann integral. In this paper we present two practical criteria of generalized Riemann integrability – one for the case of unbounded domain and another for the case of unbounded function. They cover the most relevant situations encountered in mathematical analysis and in the theory of partial differential equations.

Key words: *multidimensional generalized Riemann integral, absolute integrability, elementary solution of Laplace operator*

Introduction

In a previous paper ([6]) we discussed a definition of the multidimensional generalized (or improper) Riemann integral which covers both the case of unbounded functions and that of unbounded domains. We proved that integrability (in the sense of this definition) and absolute integrability are equivalent. Therefore, we can assume, in most cases, that the functions we are working with are nonnegative.

In this paper we intend to prove, starting from the definition given in [6,] some practical criteria of generalized integrability. Important examples are covered, such as the local integrability of the elementary solution of the Laplacian.

Again we have to mention that everything the present paper discusses is only an attempt at systematizing and proving some known facts by using relatively elementary notions.

In the first section we shall shortly recall the definition of the generalized integral and we shall reformulate the limitation criterion. In the second and in the third one we shall discuss the cases of unbounded domain and unbounded function.

Preliminaries

Definition 1. Let $B \subset \boldsymbol{R}^n$ be a locally Jordan measurable set, f a real function defined on B. We say that f is integrable in the generalized sense on B if f_M is integrable for any positive M on any Jordan subset of B and if there exists a real number I with the property that

$$(\forall)\varepsilon > 0, (\exists)A_\varepsilon \in \mathrm{J}_n, A_\varepsilon \subset B, (\exists)M_\varepsilon > 0$$

so that

$$\left|\int_A f_M(x)\mathrm{d}x - I\right| < \varepsilon, (\forall)A \in \mathrm{J}_n,\ A_\varepsilon \subset A \subset B, (\forall)M \ge M_\varepsilon .$$

Here, f_M is defined through the formula:

$$f_M(x) = -M \quad \text{if } f(x) < -M,$$
$$f_M(x) = f(x) \quad \text{if} -M \le f(x) \le M$$

and

$$f_M(x) = M \quad \text{if } f(x) > M$$

and J_n denotes the family of Jordan measurable sets.

If f is integrable in the generalized sense on B, then the number I which has the property from Definition 1 is unique, it is called the generalized or improper integral of f on B and is denoted with $\int_B f(x)\mathrm{d}x$.

If f is integrable in the generalized sense on B , then the following lemma can be used for the computation of $\int_B f(x)\mathrm{d}x$.

Lemma 1. Let f be an integrable function in the generalized sense on a locally Jordan measurable set B.

i) If

$$A_j = B \cap B(0, j),\ B(0, j) = \{x; \|x\| < j\}$$

and $M_j \to \infty$ when $j \to \infty$, then

$$\lim_{j\to\infty} \int_{A_j} f_{M_j}(x)\mathrm{d}x = \int_B f(x)\mathrm{d}x .$$

ii) If

$$A_\varepsilon = B \cap B(0,1/\varepsilon)$$

and $M_\varepsilon \to \infty$ when $\varepsilon \to 0$, then

$$\lim_{\varepsilon\to 0} \int_{A_\varepsilon} f_{M_\varepsilon}(x)\mathrm{d}x = \int_B f(x)\mathrm{d}x .$$

The proof of the lemma is quite simple: the proof of i), e.g., follows from the fact that if $\varepsilon > 0,\ A_\varepsilon,\ M_\varepsilon > 0$ are as in Definition 1, then we can find n_ε such that

$$A_n \supset A_\varepsilon,\ M_n > M_\varepsilon,\ (\forall) n \geq n_\varepsilon .$$

The following generalization of the limitation criterion will be used in the next sections.

Proposition 1. Let B be a locally Jordan measurable subset of $\boldsymbol{R}^n$ $f : B \to [0,\infty)$ a function which has the property that f_M is integrable for every positive M on every Jordan subset of *B*. Then the following assertions are equivalent

i) *f* is integrable in the generalized sense on *B*;

ii) there exists a constant $C > 0$ so that

$$\int_A f_M(x)\mathrm{d}x \leq C;$$

for every $M > 0$ and for every Jordan measurable subset *A* of *B*.

iii) if $(M_\varepsilon)_{\varepsilon>0}$ is such that $M_\varepsilon \to \infty$ when $\varepsilon \to 0$ and if $A_\varepsilon = B \cap B(0,1/\varepsilon)$, then there exists a constant $C > 0$ so that

$$\int_{A_\varepsilon} f_{M_\varepsilon}(x)\mathrm{d}x \leq C,\ (\forall)\varepsilon > 0;$$

iv) if $M_j \to \infty$ when $n \to \infty$ and if $A_j = B \cap B(0, j)$, then there exists a constant $C > 0$ so that

$$\int_{A_j} f_{M_j}(x)\mathrm{d}x \leq C,\ (\forall) j > 0.$$

Proof. The fact that i) is equivalent with ii) was already proved in [6] (Proposition 2). Also, it is quite obvious that ii) implies iii) and iii) implies iv). The implication iv) $\Rightarrow$ ii) follows from the fact that for every $M > 0$ and every Jordan measurable subset A of B there exists a natural number *n* so that $M < M_j$ and $A \subset A_j$ and *f* takes only nonnegative values.

Remark 1. i) The requirement that *f* should take only nonnegative values can be relaxed. The conclusion generated by Proposition 1 remains true if we assume that $f(x) \geq 0$ for all x in $B \setminus N$, where *N* is a Lebesgue negligible subset of *B*.

ii) If *f* is bounded from above, then one can easy see that *f* is integrable in the generalized sense on *B* if and only if there exists a constant $C > 0$ so that

$$\int_{A_j} f(x)\mathrm{d}x \leq C,\ (\forall) j > 0$$

for $A_j = B \cap B(0, j)$.

The Case of Unbounded Domains

We shall start by giving a very important and well-known example. Assume that *B* is an unbounded set of $\boldsymbol{R}^n$ which is locally Jordan measurable and such that 0 is not in the closure of B. Let

$$f_\alpha : B \to \boldsymbol{R},\ f_\alpha(x) = \|x\|^{-\alpha},\ (\forall) x \in B .$$

Then f_α is integrable in the generalized sense if $\alpha > n$.

In order to verify this assertion, we shall use Remark 1, ii). If $\varepsilon > 0$ is such that $B \cap B(0,\varepsilon) = \emptyset$ we have

$$\int_{A_j} f_\alpha(x)\mathrm{d}x \le \int_{\varepsilon^2 < \|x\|^2 < j^2} \|x\|^{-\alpha}\,\mathrm{d}x = \int_\varepsilon^j \int_{S^{n-1}} r^{-\alpha+n-1}\mathrm{d}s_n\mathrm{d}r =$$

$$= \frac{\sigma_n}{\alpha - n}\left(\varepsilon^{-\alpha+n} - j^{-\alpha+n}\right) \le \frac{\sigma_n \varepsilon^{-\alpha+n}}{\alpha - n}.$$

We denoted with σ_n, as usually, the area of the unit sphere S^{n-1}.

This example can be used to give a practical criterion of absolute integrability.

Proposition 2. If $f : B \to \boldsymbol{R}$, B a locally Jordan measurable unbounded subset of $\boldsymbol{R}^n$, is a function which is integrable on every Jordan measurable subset of B and if there exist constants $\alpha > n$, $C > 0$, $R > 0$ and N a Lebesgue negligible subset of B such that

$$|f(x)| \le C\|x\|^{-\alpha},\ (\forall)x \in B \setminus N, \|x\| > R,$$

then f is absolutely integrable..

Proof. The result follows from the comparison criterion and from the above presented example, if we consider that it is sufficient to prove the absolute integrability of f on the set

$$B \cap \left(\boldsymbol{R}^n \setminus B(0,R)\right).$$

We can also state a nonintegrability criterion.

Proposition 3. If *,* B is a locally Jordan measurable unbounded subset of $\boldsymbol{R}^n$ with the property that

$$\lim_{j\to\infty} \int_{A_j \cap \{x;\|x\|\ge 1\}} f_n(x)\mathrm{d}x = \infty$$

and if $f : B \to \boldsymbol{R}$, is a function which is integrable on every Jordan measurable subset of B and with the property that there exist constants $\alpha \le n$, $c > 0$, $R > 0$ and N a Lebesgue negligible subset of B so that

$$|f(x)| \ge c\|x\|^{-\alpha},\ (\forall)x \in B \setminus N, \|x\| > R,$$

then f is not (absolutely) integrable on B.

Proposition 3 is also a direct consequence of the hypothesis and of the comparison criterion. Among the Jordan locally measurable B sets, which satisfy the hypotheses from Proposition 3, there are sets B which contain a set of the form $C \cap \{x; \|x\| \ge R\}$, where C is an open cone with vertex in the origin and R is a positive number.

We can check this by passing in polar coordinates (the intersection of the cone with the unit sphere has positive (surface) measure).

Let us consider $C \cap S^{n-1}$ the intersection of the cone with the unit sphere.

According to Proposition 3, we shall prove that $\int_{C \cap \boldsymbol{R}^n \setminus B(0,R)} \|x\|^{-n} dx$ is nonintegrable.

Indeed,

$$\int_B \|x\|^{-n} dx \geq \int_{C\cap \boldsymbol{R}^n \setminus B(0,R)} \|x\|^{-n} dx = \int_R^{\infty} \int_{C\cap S^{n-1}} r^{-n+n-1} ds dr = \left(\int_{C\cap S^{n-1}} ds \right)\left(\int_R^{\infty} r^{-1} dr \right)$$

$$= m\left(C \cap S^{n-1}\right) \ln r\Big|_R^{\infty} \to \infty \text{, because } m\left(C \cap S^{n-1}\right) > 0;$$

where m is the surface measure.

In particular, a set *B* which contains the exterior of a ball centred in the origin, will satisfy the hypothesis of Proposition 3.

The Case of Unbounded Functions

In order to accommodate the formulation of our results within the one used when working with the Lebesgue integral, we shall slightly generalize Definition 1.

Definition 1′. Let $B' \subset \boldsymbol{R}^n$ be a locally Jordan measurable set, *N* a Lebesgue negligible set, $B = B' \cup N$ and *f* a real function defined on *B*. We say that *f* is integrable in the generalized sense on *B* if *f* is integrable in the generalized sense on *B′*. In this case the generalized integral of *f* on *B* is equal to the generalized integral of *f* on *B′*.

In this case the most important example is the following: assume that *B* is a neighbourhood of the origin. We may consider, without loss of generality, that $B = B(0, R)$ for some $R > 0$. Let us put $B = B(0, R) \setminus \{0\}$, $N = \{0\}$. Let

$$f_\alpha : B \to \boldsymbol{R},\ f_\alpha(x) = \|x\|^{-\alpha},\ (\forall) x \in B'.$$

Then f_α is integrable in the generalized sense if $\alpha < n$.

In order to verify this assertion, we shall use Proposition 1, iv) . We consider

$$A_j = B',\ M_j = j^\alpha,\ (\forall) j > 0.$$

Then

$$\int_{A_j} f_{\alpha, M_j}(x) \mathrm{d}x = \int_{j^{-2} < \|x\|^2 < R^2} \|x\|^{-\alpha} \mathrm{d}x + \int_{0 < \|x\|^2 < j^{-2}} j^\alpha \, \mathrm{d}x =$$

$$= \int_{j^{-1}}^{R} \int_{S^{n-1}} r^{-\alpha+n-1} \mathrm{d}s_n \mathrm{d}r + \mathrm{vol}(B(0, j^{-1})) j^\alpha =$$

$$= \frac{\sigma_n}{n-\alpha}\left(R^{n-\alpha} - j^{\alpha-n}\right) + \frac{\sigma_n}{n} j^{\alpha-n} \leq \frac{\sigma_n}{n-\alpha} R^{n-\alpha} + \frac{\sigma_n}{n}.$$

Therefore f_α is integrable in the generalized sense on *B′* and, according to Definition 1′, on *B*. In particular the elementary solution of the Laplace operator in $\boldsymbol{R}^n$, $n > 2$,

$$E_n = -\frac{1}{\sigma_n (n-2)} \|x\|^{2-n},\ x \neq 0$$

and its partial derivatives of first order are absolutely integrable on $B(0, R)$.

Proposition 4. If $f : B \to \boldsymbol{R}$, *B* a Jordan measurable subset of $\boldsymbol{R}^n$ whose closure contains the origin, is a function with the property that f_M is integrable for any positive *M* on $B \setminus \{0\}$ and if there exist constants $\alpha < n$, $C > 0$, $r > 0$ and *N* a Lebesgue negligible subset of $B \setminus \{0\}$ such that

$$|f(x)| \le C\|x\|^{-\alpha},\ (\forall)x \in B \setminus (N \cup \{0\}), \|x\| < r,$$

then *f* is absolutely integrable on *B*.

Proof. The result follows from the comparison criterion and from the example from above, if we consider that it is sufficient to prove the absolute integrability of *f* on the set $B \cap B(0,r)$.

A nonintegrability criterion for the case of unbounded functions is the following.

Proposition 5. If , *B* is a Jordan measurable subset of $\boldsymbol{R}^n$ whose closure contains the origin and with the property that

$$\lim_{\varepsilon \to 0} \int_{B \cap \{x; \|x\| \ge \varepsilon\}} f_n(x)\mathrm{d}x = \infty$$

and if $f : B \to \boldsymbol{R}$, is a function with the property that f_M is integrable for any positive *M* on *B* \ {0} and if there exist constants $\alpha \ge n$, $c > 0$, $r > 0$ and *N* a Lebesgue negligible subset of *B* such that

$$|f(x)| \ge c\|x\|^{-\alpha},\ (\forall)x \in B \setminus N, \|x\| < r,$$

then *f* is not (absolutely) integrable on *B*. As Proposition 3 was, Proposition 5 is also a direct consequence of the hypothesis and of the comparison criterion. Among the Jordan measurable sets *B* which satisfy the hypothesis of Proposition 5 there are sets *B* which contain a set of the form $C \cap \{x; \|x\| \le r\}$, where *C* is an open cone with vertex in the origin and *r* is a positive number. The proof is similar to the one given at the end of the previous section.

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Integrala Riemann Generalizată Multidimensională. Criterii de Integrabilitate

Rezumat

Într-o lucrare anterioară (M. Pascu, On the definition of the multidimensional Riemann integral, Buletinul Universităţii Petrol-Gaze din Ploieşti, vol. LVIII, No. 2/2006, pp. 9-16) am propus o definiţie generală a integralei Riemann generalizate. În această lucrare prezentăm două criterii practice de integrabilitate Riemann generalizată – unul pentru cazul domeniului nemărginit, celălalt pentru cazul funcţiei nemărginite. Ele acoperă cele mai relevante situaţii întâlnite în analiza matematică şi în teoria ecuaţiilor cu derivate parţiale.