Cohen-Macaulay Graphs

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Abstract

In this article the author present two methods meant to construct Cohen-Macaulay graphs and some interesting examples and properties. This paper also presents an important property of Cohen-Macalay ring to have Cohen-Macaulay fiber.

Key words: Cohen-Macaulay graph, Cohen-Macaulay ring, edge ideal

Notation and Definitions

Let $G$ be a graph on the vertex set $V = \{v_1, \ldots, v_n\}$, $E(G)$ the edge set of $G$, $R = k[x_1, \ldots, x_n]$ the polynomial ring over a field $k$, we will often identify the vertex $v_j$ with the variable $x_j$.

Definition 1. The edge ideal $I(G)$ associated to the graph $G$ is the ideal of $R$ generated by the set of square-free monomials $x_i x_j$ such that $v_i$ is adjacent to $v_j$, that is,

$$I(G) = (\{x_i x_j \mid \{v_i, v_j\} \in E(G)\}) .$$

If all the vertices of $G$ are isolated we set $I(G) = (0)$.

Note that the non zero edge ideals are precisely the ideals of $R$ generated by square-free monomials of degree two.

Definition 2. A local ring $(R, m)$ is called Cohen-Macaulay if $\text{depth}(R) = \text{dim}(R)$. If $R$ is non local and $R_p$ is a Cohen-Macaulay local ring for all $p \in \text{Spec}(R)$, then we say that $R$ is a Cohen-Macaulay ring.

Definition 3. The graph $G$ is said to be Cohen-Macaulay over the field $k$ if $R / I(G)$ is a Cohen-Macaulay ring.

Definition 4. Let $G$ be a graph with vertex set $V$. A subset $A \subseteq V$ is a minimal vertex cover for $G$ if:

1. every edge of $G$ is incident with one vertex in $A$, and
2. there is no proper subset of $A$ with the first property.

One of the purposes here is to show how large classes of Cohen-Macaulay graphs can be produced and to show some obstruction for a graph to be Cohen-Macaulay.
The First Construction

Let $H$ be a graph with vertex set $V(H) = \{x_1, ..., x_n, z, w\}$ and $J$ its edge ideal. Assume that $z$ is adjacent to $w$ with $\deg(z) \geq 2$ and $\deg(w) = 1$. We label the vertices of $H$ such that $x_1, ..., x_k, w$ are the vertices of $H$ adjacent of $z$, as shown in the figure 1.

![Fig. 1. The first construction](image)

The next two results describe how the Cohen-Macaulay property of $H$ relates to that of the two subgraphs $G = H \setminus \{z, w\}$ and $F = G \setminus \{x_1, ..., x_k\}$. One has the equalities:

$$J = (I, x_1z, ..., x_kz, zw) \quad \text{and} \quad (I, x_1, ..., x_k) = (L, x_1, ..., x_k),$$

where $I = I(G)$ and $L = I(F)$ are the edge ideals of $G$ and $F$ respectively.

Assume $H$ is unmixed with height of $J$ equal to $g + 1$. Since $z$ is not isolated, there is a minimal prime $p$ over $I$ containing $\{x_1, ..., x_k\}$ and such that $ht(I) = ht(p) = g$. It is not difficult to prove that $k < n$ and $\deg(x_i) \geq 2$ for $1 \leq i \leq k$.

**Proposition 1.** If $H$ is a Cohen-Macaulay graph, then $F$ and $G$ are Cohen-Macaulay graphs.

**Proof.** Let $A = k[x_1, ..., x_n]$ and $R = A[z, w]$. There exists a homogeneous system of parameters $\{f_1, ..., f_d\}$ for $A/I$, where $f_i \in A_+$ for all $i$. Because of the hypothesis and the equalities

$$z(z - w) + zw = z^2 \quad \text{and} \quad w(w - z) + zw = w^2,$$

the set $\{f_1, ..., f_d, z - w\}$ is a regular system of parameters for $R/J$.

Hence $\{f_1, ..., f_d\}$ is a regular sequence on $R/I$, that is, $G$ is Cohen-Macaulay.

Now we consider the sequence

$$0 \rightarrow R \xrightarrow{(I, x_1, ..., x_k, w)} R \xrightarrow{(I, z)} R \xrightarrow{\psi} 0,$$

where the first map is the multiplication by $z$ and $\psi$ is induced by a projection.

By the depth lemma one has $n - g + 1 \leq \text{depth} \frac{R}{(I, x_1, ..., x_k, w)}$, where $g = ht(J)$.

Since $(I, x_1, ..., x_k, w) = (L, x_1, ..., x_k, w)$, $F$ is Cohen-Macaulay graph.
**Proposition 2.** If $F$ and $G$ are Cohen-Macaulay graphs and $\{x_1, \ldots, x_k\}$ form a part of a minimal vertex cover for $G$, then $H$ is Cohen-Macaulay graph.

**Proof.** Consider the exact sequence

$$0 \rightarrow \frac{R}{(I, x_1, \ldots, x_k, w)}(-1) \rightarrow \frac{R}{J} \rightarrow \frac{R}{(I, z)} \rightarrow 0$$

Since $\frac{R}{(I, x_1, \ldots, x_k, w)}$ and $\frac{R}{(I, z)}$ are Cohen-Macaulay rings, then $\frac{R}{J}$ is Cohen-Macaulay ring, that is, $H$ is Cohen-Macaulay graph.

**Corollary 3.** If $G$ is Cohen-Macaulay graph and $\{x_1, \ldots, x_k\}$ is the minimal vertex cover for $G$, then $H$ is Cohen-Macaulay graph.

**Proof.** Since $I(F) = (0)$ result $F$ is Cohen-Macaulay graph and we apply proposition 2.

**The Second Construction**

For the discussion of the second construction we change our notation. Let $H$ be a graph on the vertex set $V(H) = \{x_1, \ldots, x_n, z\}$ so that $\{x_1, \ldots, x_k\}$ be the vertex of $H$ adjacent to $z$.

We may assume $\deg(x_i) \geq 2$ for $1 \leq i \leq k$ and $\deg(z) \geq 2$.

Setting $G = H \setminus \{z\}$ and $F = G \setminus \{x_1, \ldots, x_k\}$, notice that the ideals $J = I(H)$, $I = I(G)$ and $L = I(F)$ associated to $H$, $G$ and $F$ respectively are related by the equalities:

$$J = (I, x_1, \ldots, x_k z) \text{ and } (I, x_1, \ldots, x_k) = (L, x_1, \ldots, x_k).$$

**Proposition 4.** If $H$ is Cohen-Macaulay graph, then $F$ is Cohen-Macaulay graph.

**Proof.** Let $A = k[x_1, \ldots, x_n]$ and $R = A[z]$ and $\text{ht } J = g + 1$. The polynomial $f = z - x_1 - \ldots - x_k$ is regular on $R / J$ because it is not contained in any associated prime of $J$.

There is a sequence $\{f_1, \ldots, f_{n-g-1}\}$ regular on $\frac{R}{J}$ so that $\{f_1, f_2, \ldots, f_{n-g-1}\} \subset A_+$. Observe that $\{f_1, \ldots, f_{n-g-1}\}$ is in fact a regular sequence on $\frac{A}{I}$, which gives depth $\frac{A}{I} \geq n - g - 1$.

Now, we use the sequence

$$0 \rightarrow \frac{R}{(I, x_1, \ldots, x_k)}(-1) \rightarrow \frac{R}{J} \rightarrow \frac{R}{(I, z)} \rightarrow 0$$

and $\text{ht } (I, x_1, \ldots, x_k) = g + 1$ to conclude that $F$ is Cohen-Macaulay graph.

**Proposition 5.** Assume $x_1, \ldots, x_k$ do not form a part of a minimal vertex cover for $G$ and $\text{ht } (I, x_1, \ldots, x_k) = \text{ht } (I) + 1$. If $F$ and $G$ are Cohen-Macaulay graphs, then $H$ is Cohen-Macaulay graph.

**Proof.** The assumption on $\{x_1, \ldots, x_k\}$ forces $\text{ht } (J) = \text{ht } (I) + 1$.

From the exact sequence:
we obtain that $H$ is Cohen-Macaulay graph.

**Corollary 6.** If $G$ is Cohen-Macaulay graph and \( \{x_1, ..., x_{k-1}\} \) is a minimal vertex cover for $G$, then $H$ is Cohen-Macaulay graph.

A good property of Cohen-Macaulay graphs is its additivity with respect to connected components.

**Lemma 7.** Let $R_1 = k[x_1, ..., x_n]$ and $R_2 = k[y_1, ..., y_m]$ be two polynomial rings over a field $k$ and $R = k[x_1, ..., x_n, y_1, ..., y_m]$. If $I_1$ and $I_2$ are graded ideals in $R_1$ and $R_2$ respectively, then

\[
\text{depth } \left( \frac{R_1}{I_1} \right) + \text{depth } \left( \frac{R_2}{I_2} \right) = \text{depth } \left( \frac{R}{I_1 + I_2} \right).
\]

**Proof.** Because $\frac{R_2}{I_2} \otimes_k R_1 = \frac{R}{I_1 + I_2}$ is a flat $R_1$ module the equality holds from a general property of tensor products.

**Proposition 8.** If $G$ is a graph and $G_1, ..., G_n$ its connected components, then $G$ is Cohen-Macaulay graph if and only if $G_1, ..., G_n$ are Cohen-Macaulay graphs.

**Proof.** Evident from Lemma 7.

**References**


**Grafuri Cohen-Macaulay**

**Rezumat**

Scopul acestui articol este de a prezenta construcția unor clase de grafuri Cohen-Macaulay și câteva exemple. De asemenea, este prezentată o proprietate foarte importantă a inelelor Cohen-Macaulay și anume ca fibra să fie Cohen-Macaulay.